

NASA TECHNICAL  
MEMORANDUM

NASA TM X-53108

AUGUST 17, 1964

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NASA TM X-53108

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<b>N64-29729</b>	
(ACCESSION NUMBER)	
64	(PAGES)
<b>JMK-53108</b>	
(NASA CR OR TMX OR AD NUMBER)	
29	(THRU)
29	(ODC)
(CATEGORY)	

## CHARACTERISTIC FEATURES OF SOME PERIODIC ORBITS IN THE RESTRICTED THREE BODY PROBLEM

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ABSTRACT

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*Author*

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

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#### ACKNOWLEDGEMENTS

The author wishes to thank Dr. Richard Arenstorf and Mr. Mirt Davidson of the MSFC Computation Laboratory who by their advice and stimulation were instrumental in achieving these results. The author would also like to express his gratitude and appreciation to Mr. Thomas French for his help in generating and presenting the data contained in this report.

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SUMMARY

Earth-moon orbits are presented which, when referred to a rotating coordinate system, return periodically to their original set of state variables. Information concerning the period of the orbit, time spent in the region between earth and moon, close approach distance to the moon, and closest approach distance to the earth is given for various families of periodic orbits. These orbits have periods of 1 to 4 months, and they have at least one perpendicular crossing of the earth-moon line on the back side of the moon.

SECTION I: INTRODUCTION

This report presents earth-moon orbits which, when referred to a rotating coordinate system, return periodically to their original set of state variables. Such orbits offer repeated approaches to both earth and moon and could be used in instrumented exploration of earth-moon space for meteoroid concentration and radiation belts.

Information concerning the period of the orbits, time spent in the region between earth and moon, close approach distance to the moon, and closest approach distance to the earth is given for various families of periodic orbits. The orbits discussed in this paper have periods of 1 to 4 months, and they have at least one perpendicular crossing of the earth-moon line on the back side of the moon. The orbits contained herein represent only a small portion of the families of periodic orbits that are possible in the restricted three body problem, and it should not be inferred that these are the only orbits of interest.

## SECTION II: DISCUSSION

### A. EXISTENCE

The existence of certain periodic orbits in the restricted three body problem has been known for a long time. Poincare' referred to orbits which reduce to circles when the disturbance from the more distant body becomes zero as "Solutions de la premiere sorte."<sup>1</sup> These orbits can be near either of the finite masses, but not both. Arenstorf<sup>2</sup> proved the existence of periodic solutions of the so-called second kind which are near rotating Keplerian ellipses. Orbits of the first and second kind exist even if one of the masses becomes massless.

Contrary to this, there are periodic orbits that exist only in the restricted three body problem proper. In the earth-moon system such orbits would owe their existence to the disturbance produced by the moon; however, they degenerate into orbits of the second kind when the disturbance by the moon becomes zero. This report presents periodic orbits of the second kind and orbits that are inherent in the restricted three body problem proper.

### B. BASIC ASSUMPTIONS

A restricted three body model is assumed for the earth, moon, and probe system. In this system, the earth and moon revolve in circles in a common plane around their common center of mass (barycenter). For this investigation, the probe's motion is restricted to the plane defined by the earth-moon motion. The equations of motion are normalized such that the sum of the masses of the earth and moon is unity, the constant distance between the earth and moon is unity, and the period of the earth and moon about their common center of mass is  $2\pi$ .

A rotating coordinate system (origin at the barycenter) in which the earth and moon lie on the x-axis is advantageous because of image properties which occur in the system. In this system, under restricted three body assumptions, two perpendicular crossings of the earth-moon line (x-axis) are sufficient for periodicity. With this in mind, orbits were generated by starting on the back side of the moon perpendicular to the earth-moon line. This was an arbitrary choice of starting conditions, but they proved quite convenient. With one perpendicular crossing assured, the problem is to isolate transits which have a second perpendicular crossing of the earth-moon line. In this study, the velocity magnitude at the starting position (first perpendicular crossing)

was varied in order to perform the isolation. Complete families were generated by changing the starting position and repeating the process. Orbits that are retrograde as they approach the earth are neglected. Future investigations are planned in this area and should add insight to the general behavior of periodic orbits.

The ratio of the mass of the earth to the mass of the moon was assumed to be 80.45, and for the purposes of converting from the unitized system to a physical system, the distance from the center of the earth to the center of the moon was taken to be 385,000 km.

### C. CLASSIFICATION

The classification of orbit families used in this report is the same as the system used by Arenstorf<sup>3</sup> and Davidson\*. Categories such as ratio, order, and class are used in distinguishing various families of orbits. These terms will be used extensively; therefore, a brief explanation of each is in order.

Figures 1 and 2 depict a periodic orbit in a rotating and a space fixed frame of reference, respectively. One notices in the space fixed system that the probe makes two revolutions in its orbit in the same time the moon makes approximately one revolution in its orbit. The major axis of the probe's orbit has been rotated slightly due to the disturbance by the moon; therefore, the period of the orbit is less than the period of the moon, and the orbit is not closed in the space fixed frame of reference. However, closure in the space fixed frame of reference is not necessary for periodicity in the restricted three body problem. If one lets  $m$  equal the number of revolutions the moon makes while the probe has to make  $k$  revolutions in its orbit before periodicity occurs, then the ratio  $m/k = \frac{1}{2}$  is used to classify this orbit.

Kepler's third law provides an estimate for a minimum value of  $m/k$  for orbits that encompass both the earth and moon. In the unitized coordinate system, the period of the moon is given as  $P_M = 2\pi$ , and the period of the probe about the earth is

$$P_p = 2\pi\sqrt{a^3}$$

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\*Private communication with M. C. Davidson of the Computation Laboratory, MSFC.

where  $a$  is the semimajor axis of the probe's orbit. If the probe's orbit is to contain both masses, then  $a \geq \frac{1}{2}$ . Under this assumption, the minimum value of  $m/k$  is

$$\left(\frac{1}{2}\right)^{\frac{3}{2}} \approx .354.$$

Figure 1 is an orbit of ratio  $\frac{1}{2}$  order 1. Orbit with ratio  $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \dots$  will be referred to as higher order orbits (second, third, fourth ...) of ratio  $\frac{1}{2}$ . In general, an orbit with a ratio  $\frac{nm}{nk}$  is classified as ratio  $\frac{m}{k}$  order  $n$ . Figures 3 and 4 show orbits of ratio  $\frac{1}{2}$  order 2. In Figure 3 the second perpendicular crossing of the earth-moon line occurs on the back side of the moon. Orbit with this characteristic are designated class A. In Figure 4 the second perpendicular crossing occurs on the front side of the moon, and orbits of this nature will be referred to as class B. First order orbits are periodic orbits of the second kind, but the higher order orbits are orbits of the restricted three body problem proper.

All orbits presented in this report have at least one perpendicular crossing on the back side of the moon. There are orbits which do not possess this characteristic, but complete data on these orbits are not available at this time.

#### D. APPLICATIONS

A knowledge of the conditions that exist in earth-moon space is desirable prior to manned lunar missions. Information about meteoroid concentration, radiation belts, and other useful data could be obtained by a long life (1 year or more) unmanned spacecraft in an earth-moon periodic orbit. A desirable periodic orbit would provide adequate mapping of the region of earth-moon space traversed by an Apollo type trajectory. Therefore, in choosing an orbit, one should consider the amount of time spent (coverage) in the desired region.

In the restricted three body problem, the moon's orbit is assumed to be circular. In the true physical system the moon's orbit is near-elliptical, and a velocity budget will be required for orbit keeping. However, one can, in limited cases, overcome this perturbation by employing periodic orbits with periods that are exact multiples of the moon's period. Orbit of this nature will be closed in both the rotating and space fixed frame of reference, and will, after a given period of time, return to their original state.

It is seen in Figures 3 and 4 that the close approaches to the earth usually occur in pairs. For example, in Figure 3 there are four close approaches to the earth, and the maximum difference in altitude between the two pairs is 1,240 km. These close approaches provide four chances for orbit injection.

### SECTION III: RESULTS

Families were studied by varying the starting position behind the moon, and the results are presented with this as the independent parameter. Closest approach distance to the center of the earth, the period of the orbit, and per cent time on the inner leg of the orbit are presented for various families. The inner leg of an orbit is defined as the part of the orbit that lies closest to the earth-moon line and extends from perigee at the earth to perisel at the moon and back to perigee at the earth. The second approach to the moon and the difference between the maximum and minimum perigee altitudes ( $\Delta R$ ) at the earth are presented also. Data for orbits of ratio  $\frac{1}{2}$  order 1 are presented in Figure 5. Orbita of this family exist for starting positions (perpendicular crossing on the back side of the moon) ranging from the moon's surface out to a radius of 89,400 km. At radii slightly greater than 89,400 km, the orbits impact the surface of the earth, and a further increase in the starting radius produces retrograde orbits.

Figures 6, 7, and 8 show data for ratio  $\frac{1}{2}$  order 2 class B. These orbits exist for starting radii between 3,075 km and 57,000 km. Collision with the surface of the earth occurs for starting radii less than 3,075 km and greater than 57,000 km. It is evident from Figure 7 that this family contains an orbit with a period of  $4\pi$  (2 lunar months). This indicates that this family contains an orbit that will be periodic even when the moon's orbit is assumed to be elliptic.

Depicted in Figures 9, 10, and 11 are data for ratio  $\frac{1}{2}$  order 2 class A. Collision with the surface of the earth occurs with a starting radius of 1,928 km. For this starting radius, the second perpendicular crossing of the earth-moon line occurs on the back side of the moon at a distance of 132,000 km. As the starting radius is continuously increased, the second perpendicular crossing moves in toward the moon until the two crossings coincide. This occurs at about 15,000 km. Transits that are started beyond this radius will have their second perpendicular crossing between the moon and the starting position, and they will be duplicates of transits

that were started from a position inside the 15,000 km limit.

Data for orbits of ratio  $\frac{1}{2}$  order 3 (an example is shown in Figure 12) are presented in Figures 13, 14, and 15. Collision with the earth occurs for starting radii less than 2,310 km. As seen in Figure 12, the second perpendicular crossing for this family occurs behind the earth. The second close approach to the moon takes place after apogee, or alternatively stated, on the descending leg of the space fixed orbit. A similar family exists for this ratio and order in which the second close approach occurs on the ascending leg of the space fixed orbit. An illustration of this type of orbit is given in Figure 16; however, complete data are not available for analysis of this family.

Figures 18, 19, and 20 show data for orbits of ratio  $\frac{1}{2}$  order 4 class A. The closest approach to the center of the earth (11,895 km) occurs for a zero starting altitude at the moon. As presented in Figure 18, the second approach to the moon is the close approach that lies on the earth-moon line.

Data for orbits of ratio  $\frac{2}{3}$  order 1 are given in Figure 22. These orbits can be found with starting radii beginning at the moon's surface and extending out to 183,000 km.

Depicted in Figures 24 and 25 are data for ratio  $\frac{2}{3}$  order 2 class A. Close approach to the earth (93,610 km) occurs for the smallest possible starting radius at the moon (1,738 km). Orbit with starting radii greater than 8,900 km will have their second perpendicular crossing between the starting position and the moon and will, therefore, be a duplicate of a transit that was started at this smaller radius.

Figures 27, 28, 29, and 30 show data for orbits of ratio  $\frac{2}{3}$  order 2 class B. At a starting radius of 51,000 km, there exists an orbit with a period of  $8\pi$ . This family contains two solutions for the same starting radius for starting radii near 1,994 km and 110,000 km; however, further investigation is necessary in order to determine the exact area in which these dual solutions exist.

Data for orbits of ratio  $\frac{2}{5}$  order 1 are shown in Figures 32 and 33. This family of orbits exists for starting radii between 7,800 km and 19,800 km. Beyond these limits the orbits collide with the surface of the earth.

Information for orbits of ratio  $\frac{3}{5}$  order 1 is given in Figures 35 and 36. Members of this family were found for starting radii from 3,187 km to 74,026 km. Two solutions were found for each starting position, and the alternate solution (the solution with the highest velocity) is denoted by an asterisk. Data are given in Figures 38 and 39. The velocity difference between two orbits from the same radius varied from 6.4 m/sec to 38.4 m/sec.

Data for orbits of ratio  $\frac{3}{5}$  order 2 class B are given in Figures 41, 42, and 43. The closest approach to the earth occurs for a starting radius of 2,802 km.

Graphs giving velocity as a function of starting position for the various ratios and orders are shown in Figures 44 through 50.

#### SECTION IV: CONCLUDING REMARKS

Periodic orbits of ratio  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{2}{5}$ , and  $\frac{3}{5}$  have been investigated. Higher order orbits of these ratios are being studied as well as different ratios, and these will be described in a later paper. To aid in mission planning, Figure 51 shows a summary of some of the orbits which offer injection altitudes at the earth of approximately 100 nautical miles.

#### REFERENCES

1. Poincare, H., Les Méthodes Nouvelles de la Mecanique Céleste, Vol. I, p. 97.
2. Arenstorf, R. F., "Periodic Solutions of the Restricted Three Body Problem Representing Analytic Continuation of Keplerian Elliptic Motion," NASA TN D-1859, May 1963.
3. Arenstorf, R. F., "Periodic Trajectories Passing Near Both Masses of the Restricted Problem of Three Bodies," Presented at the XIVth International Astronautical Congress, Paris, France, September 26 - October 1, 1963.

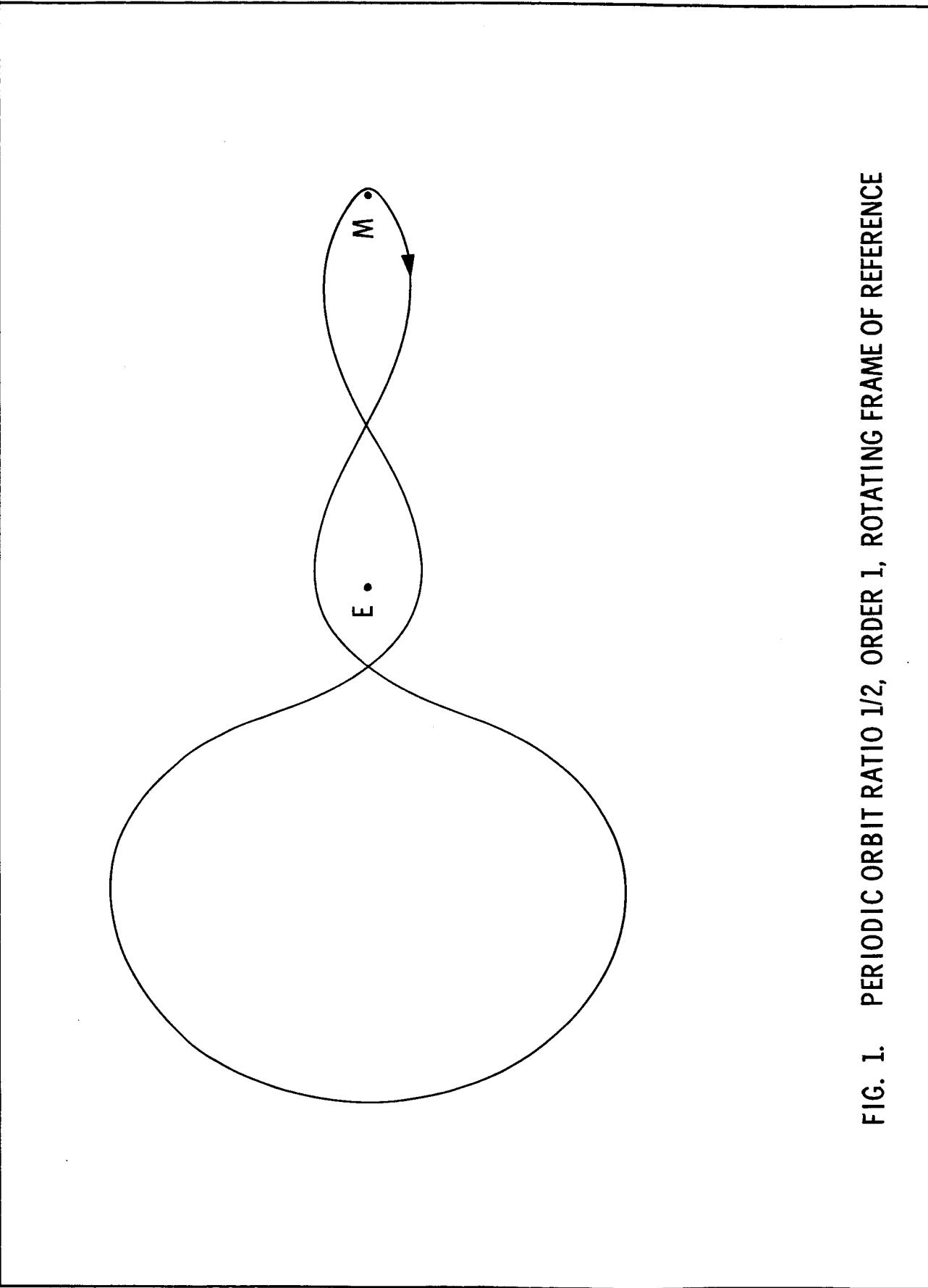


FIG. 1. PERIODIC ORBIT RATIO 1/2, ORDER 1, ROTATING FRAME OF REFERENCE

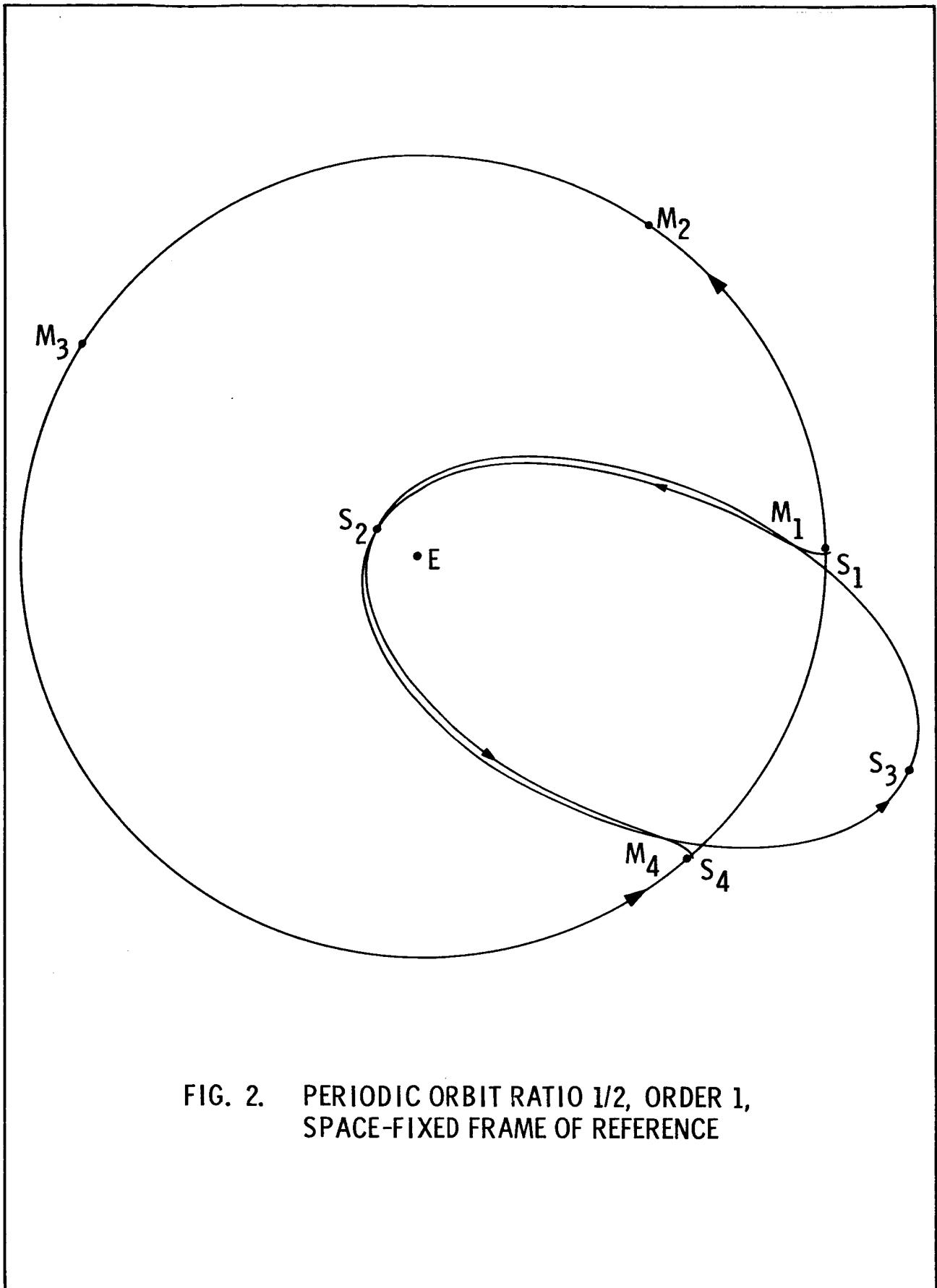


FIG. 2. PERIODIC ORBIT RATIO 1/2, ORDER 1,  
SPACE-FIXED FRAME OF REFERENCE

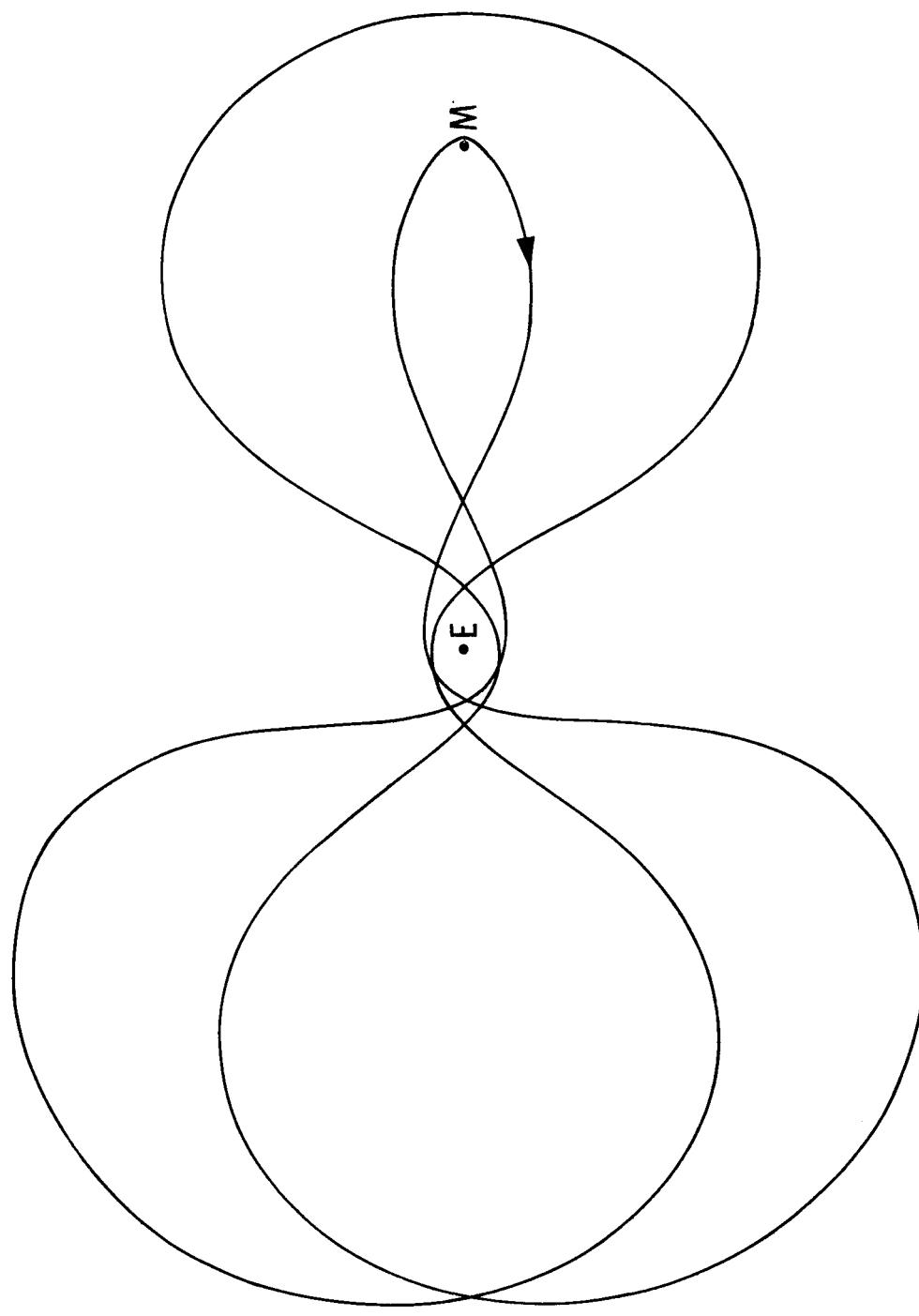


FIG. 3. RATIO 1/2, ORDER 2, CLASS A

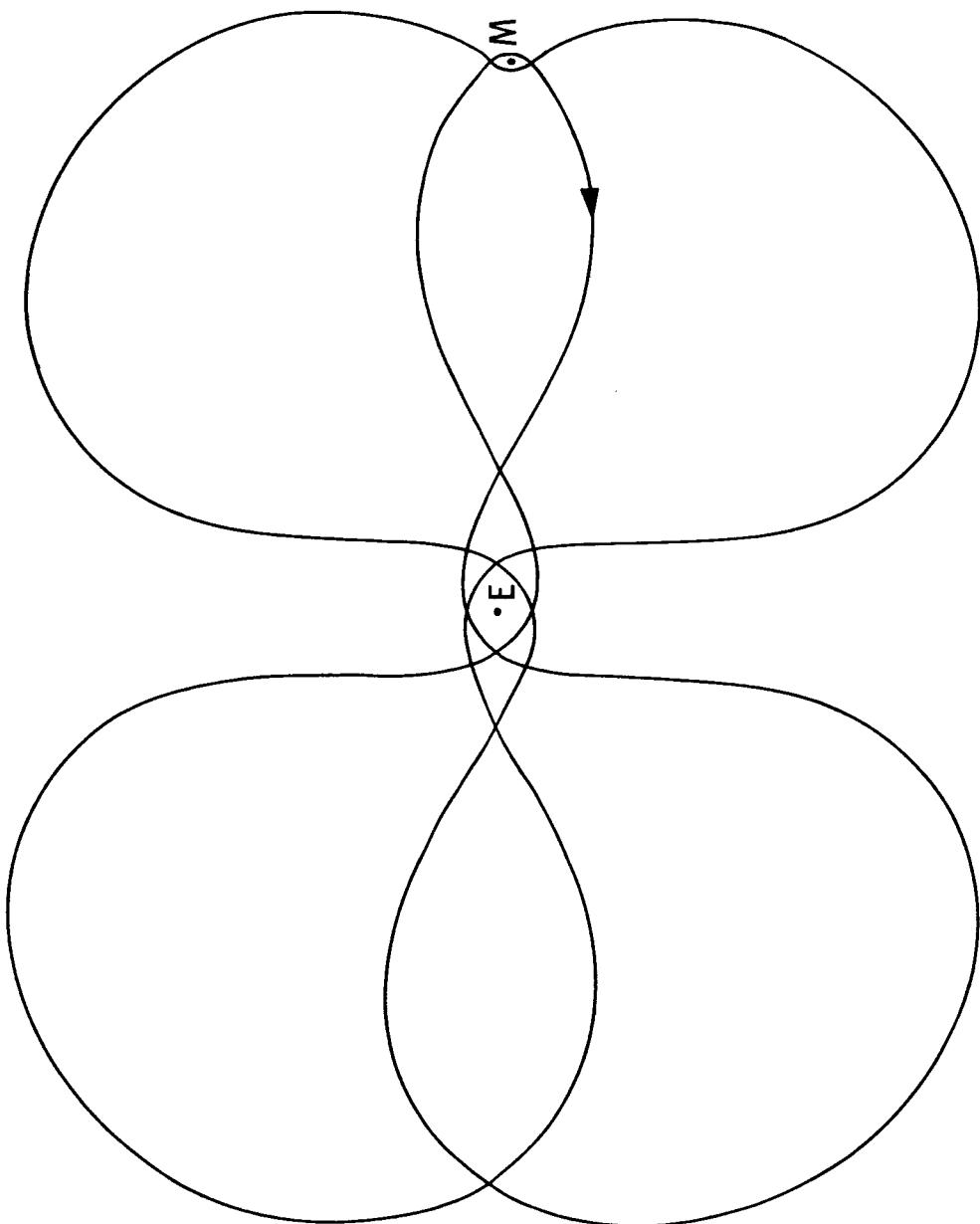


FIG. 4. RATIO 1/2, ORDER 2, CLASS B

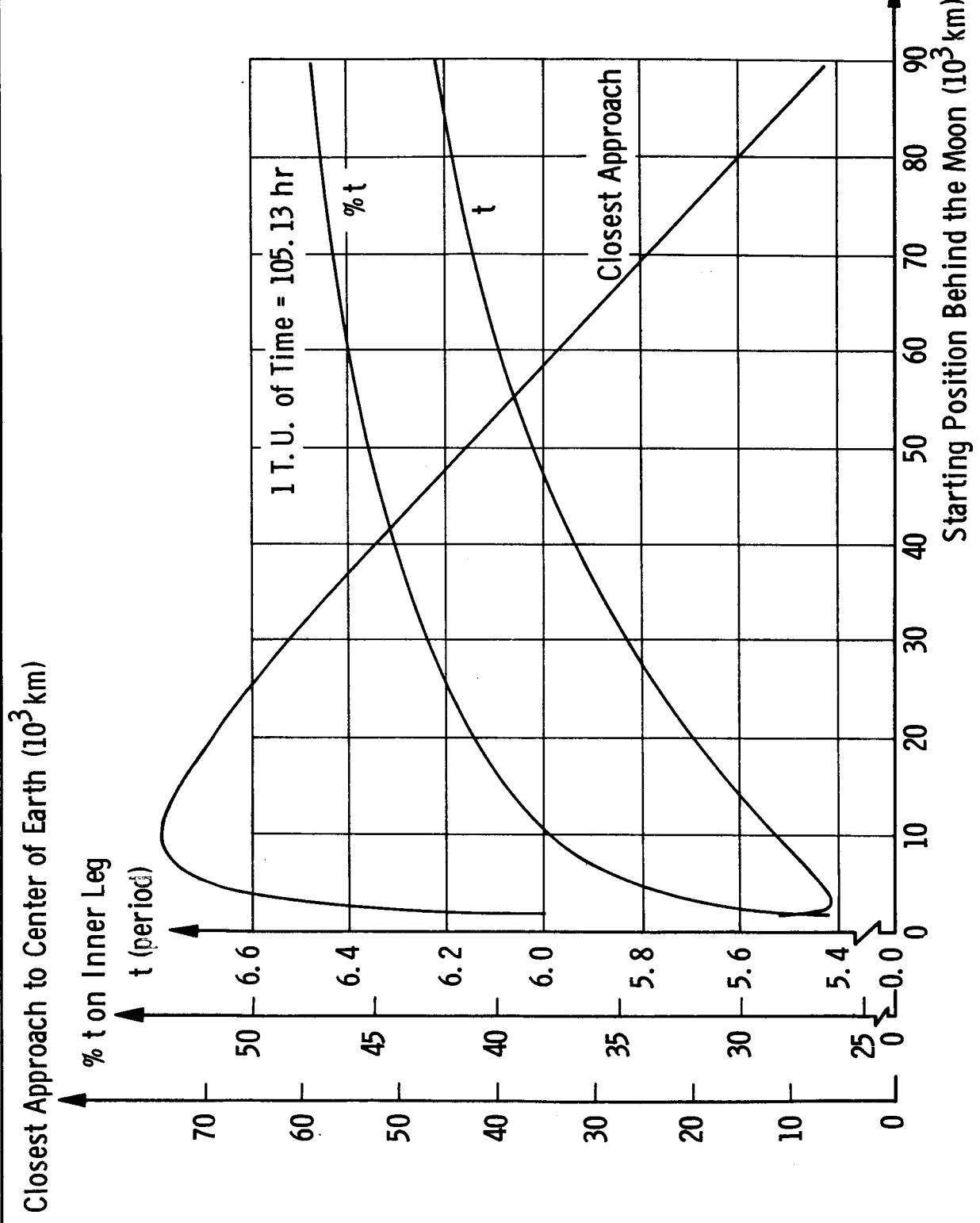


FIG. 5. RATIO 1/2, ORDER 1

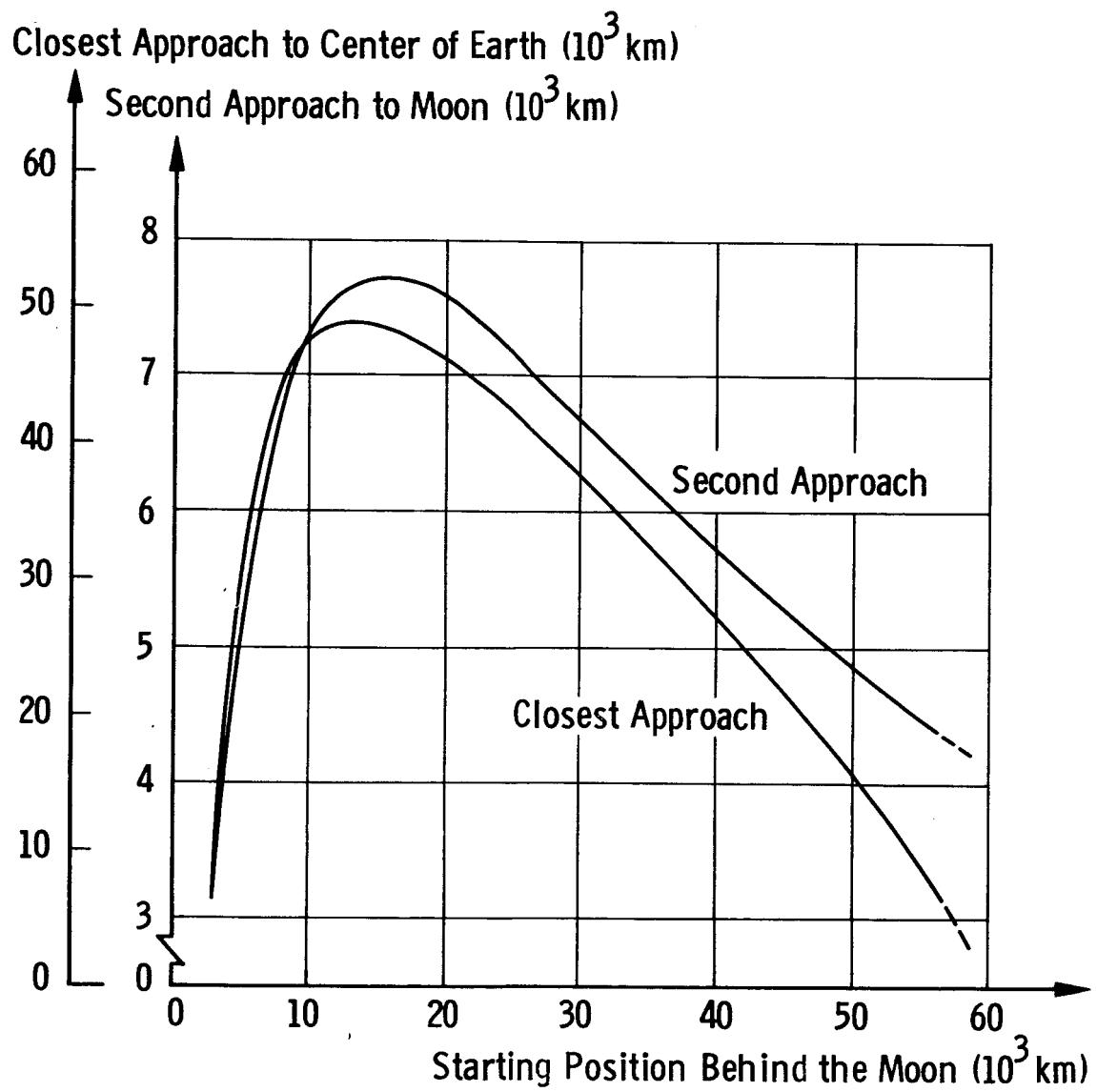


FIG. 6. RATIO 1/2, ORDER 2, CLASS B

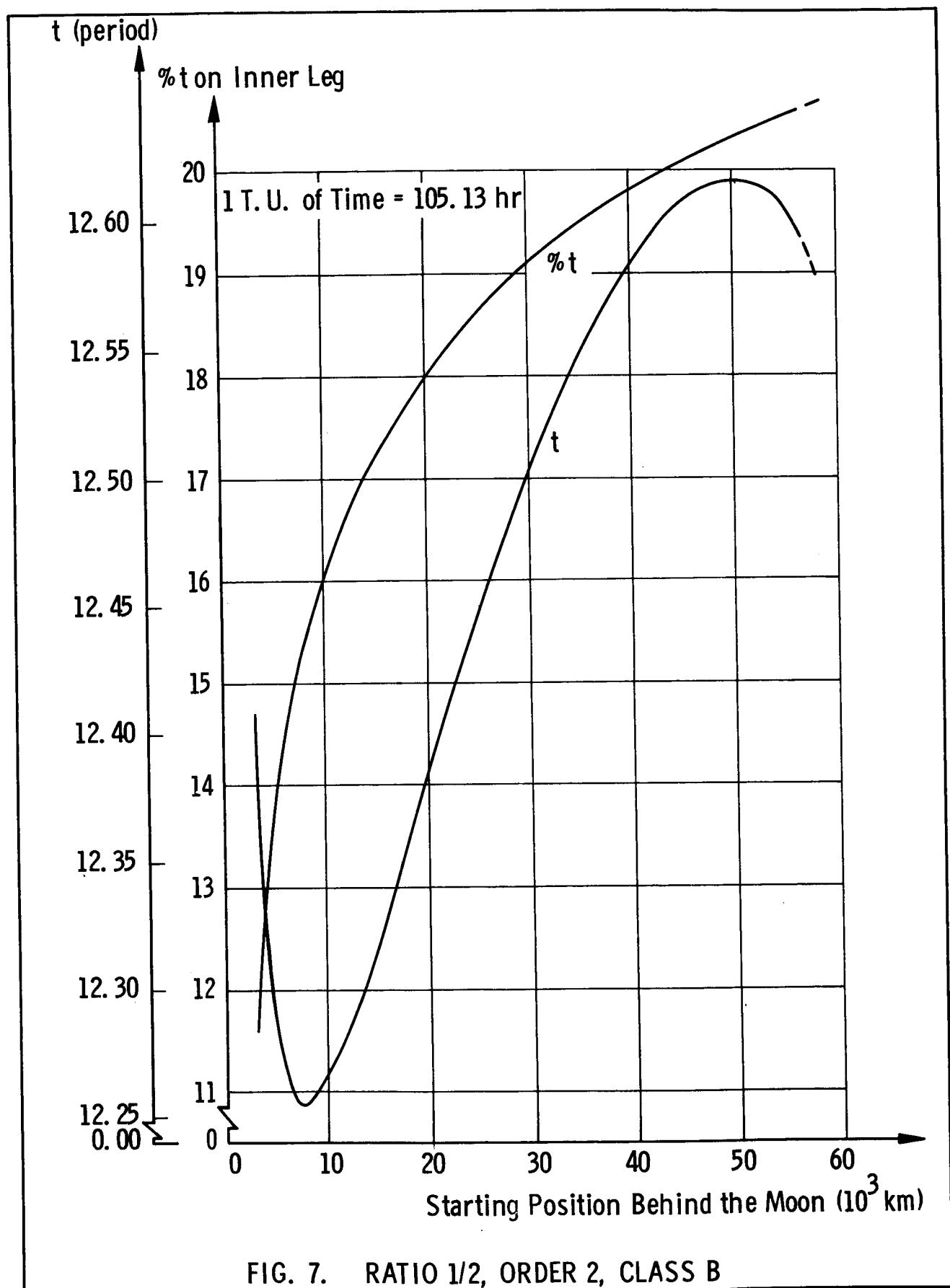


FIG. 7. RATIO 1/2, ORDER 2, CLASS B

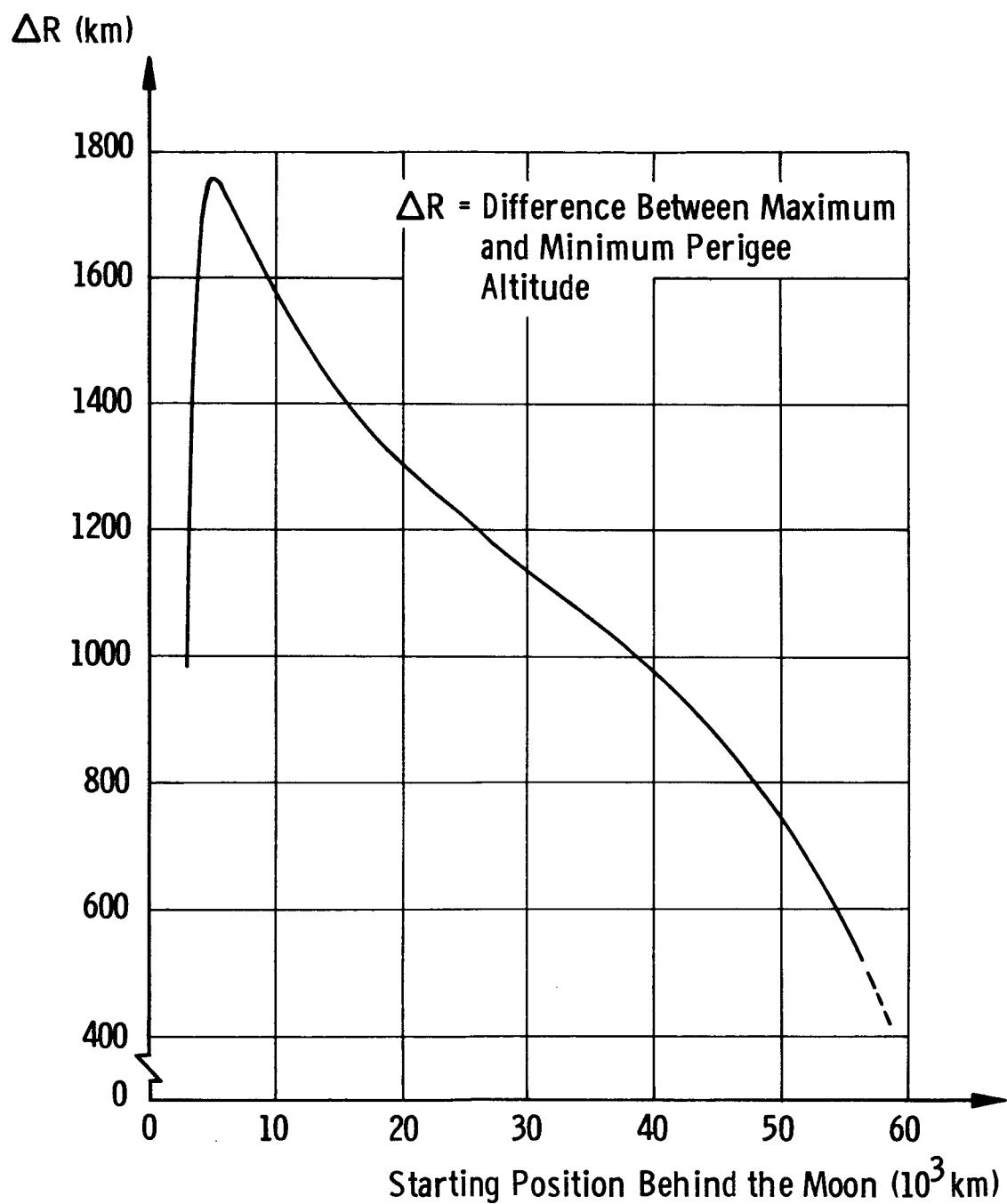


FIG. 8. RATIO 1/2, ORDER 2, CLASS B

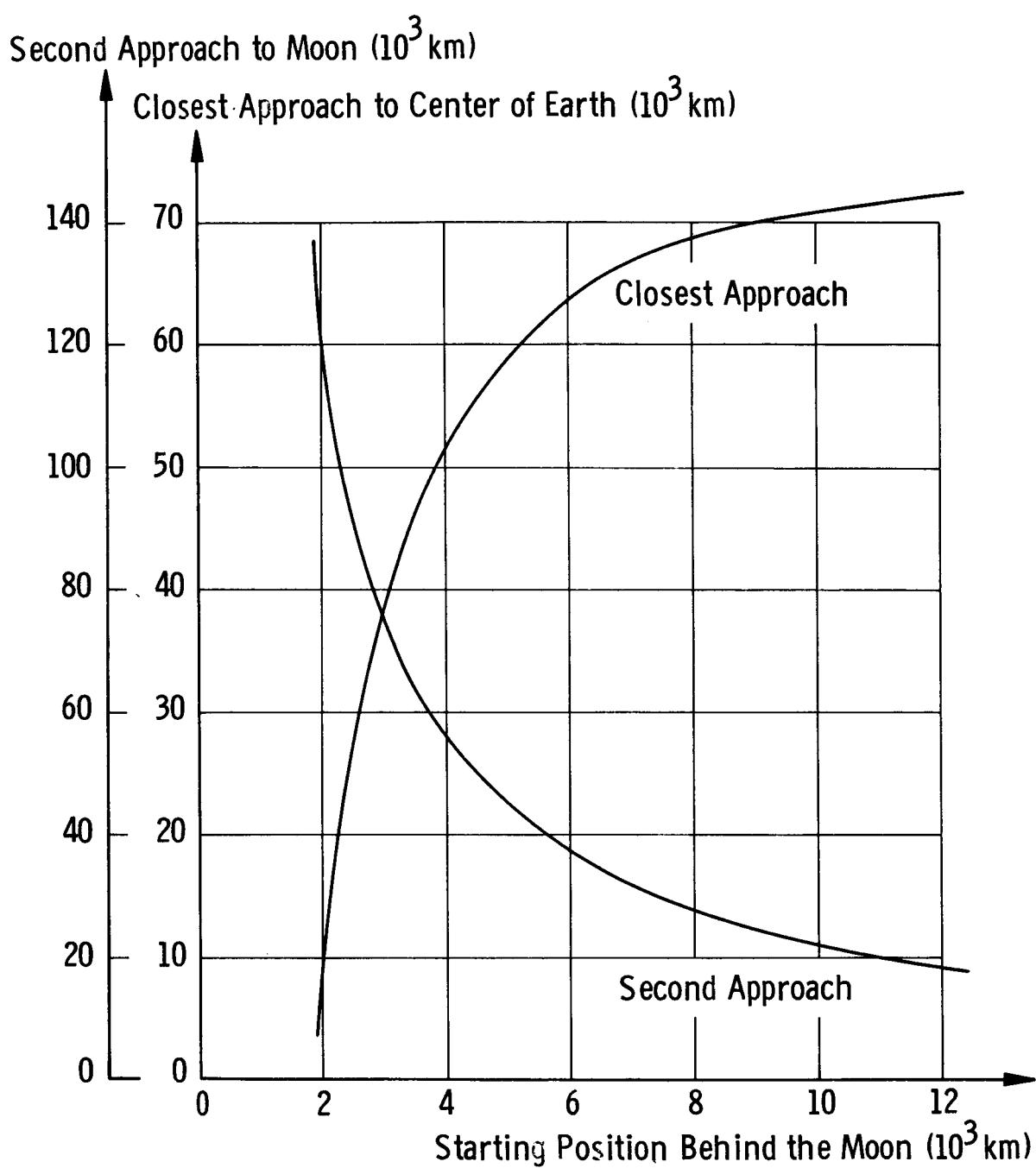
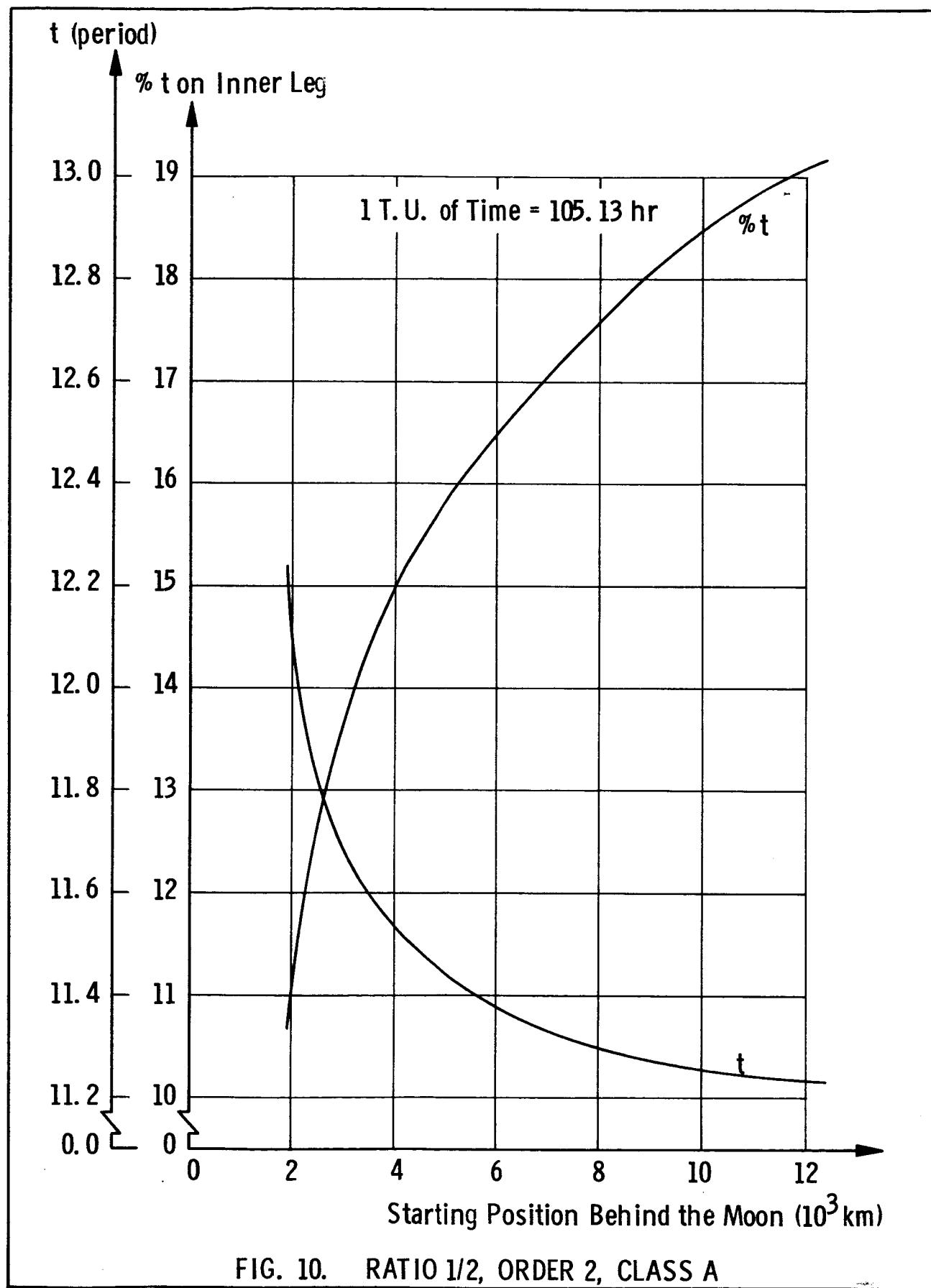


FIG. 9. RATIO 1/2, ORDER 2, CLASS A



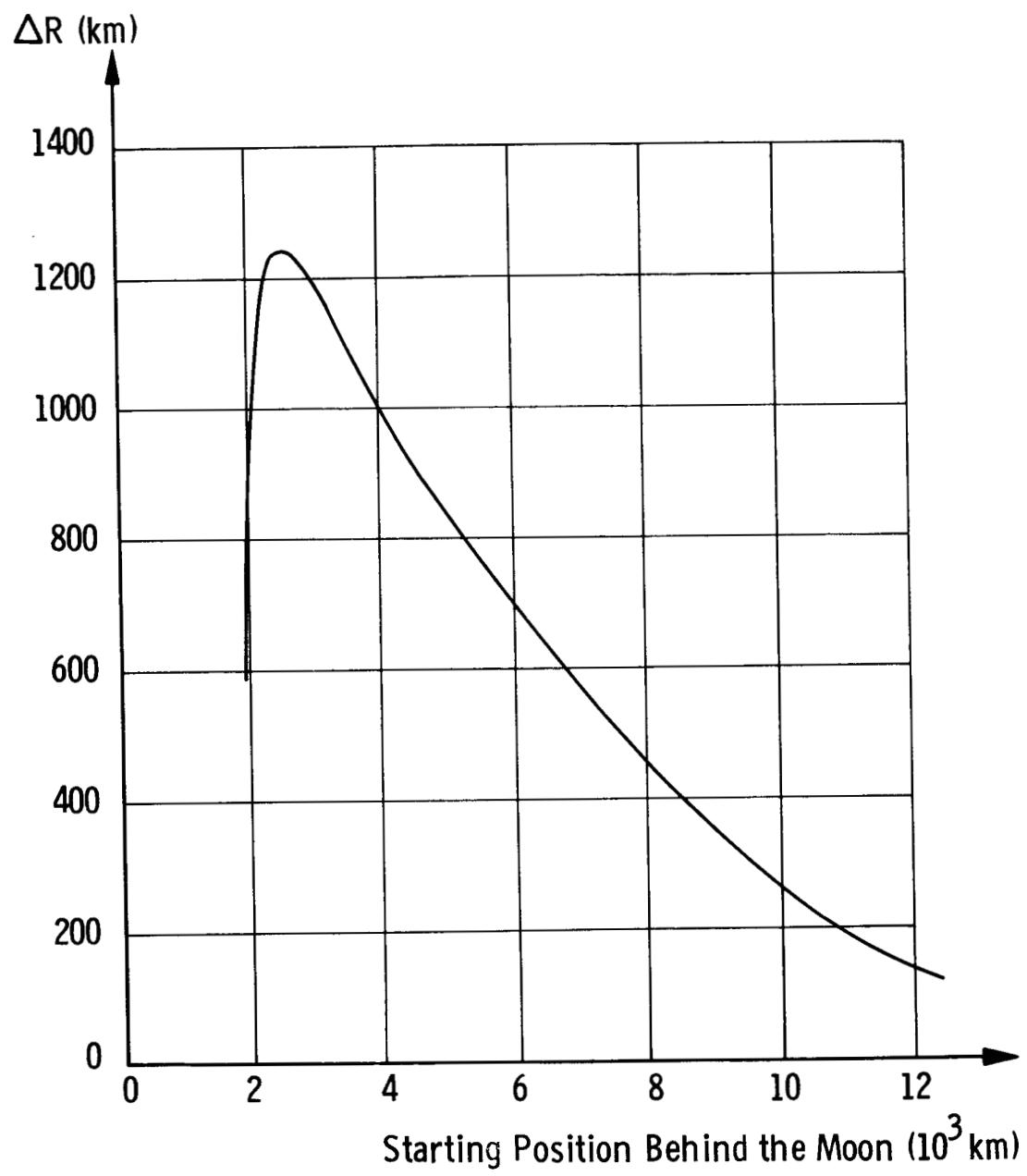


FIG. 11. RATIO 1/2, ORDER 2, CLASS A

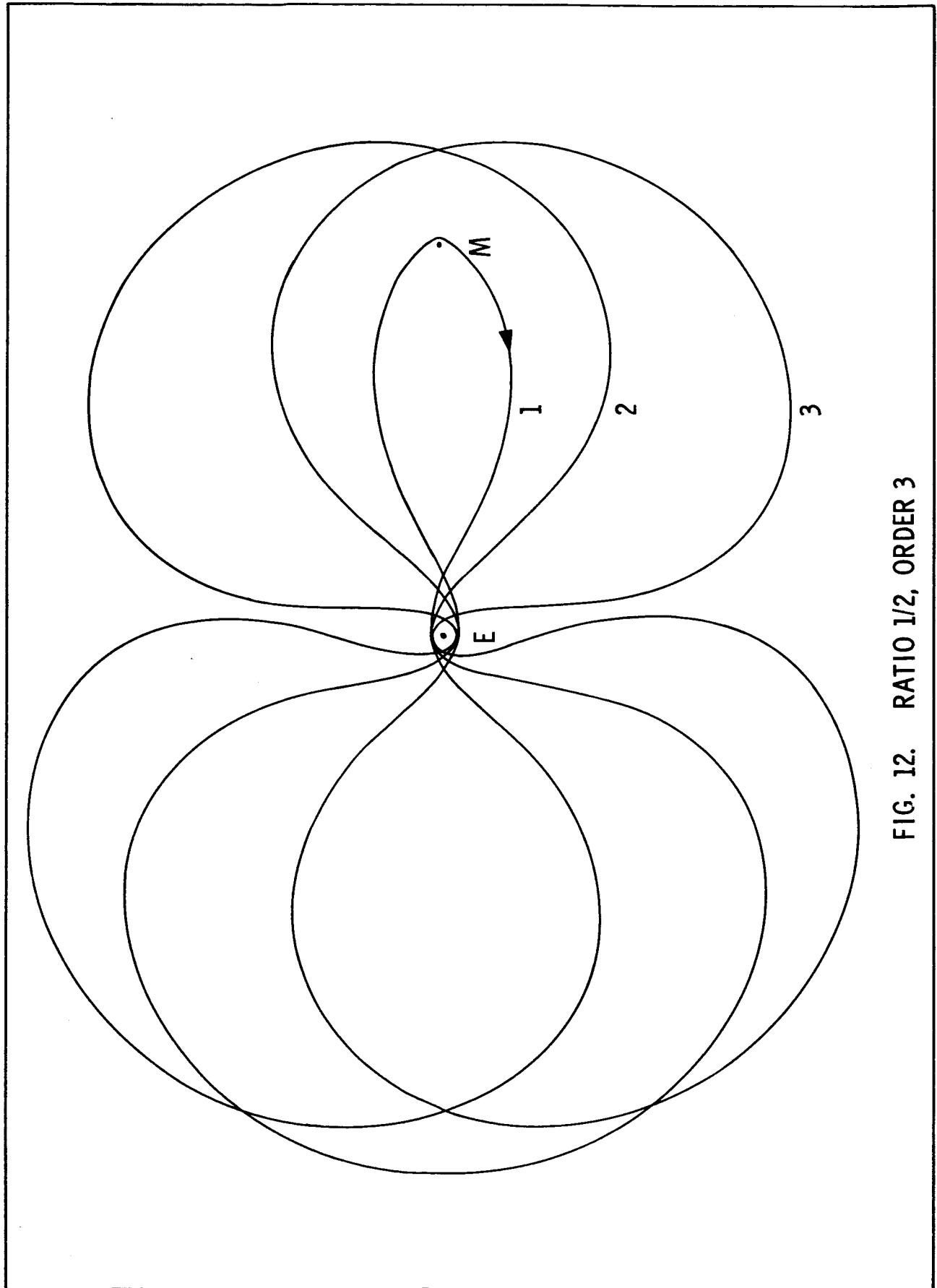


FIG. 12. RATIO 1/2, ORDER 3

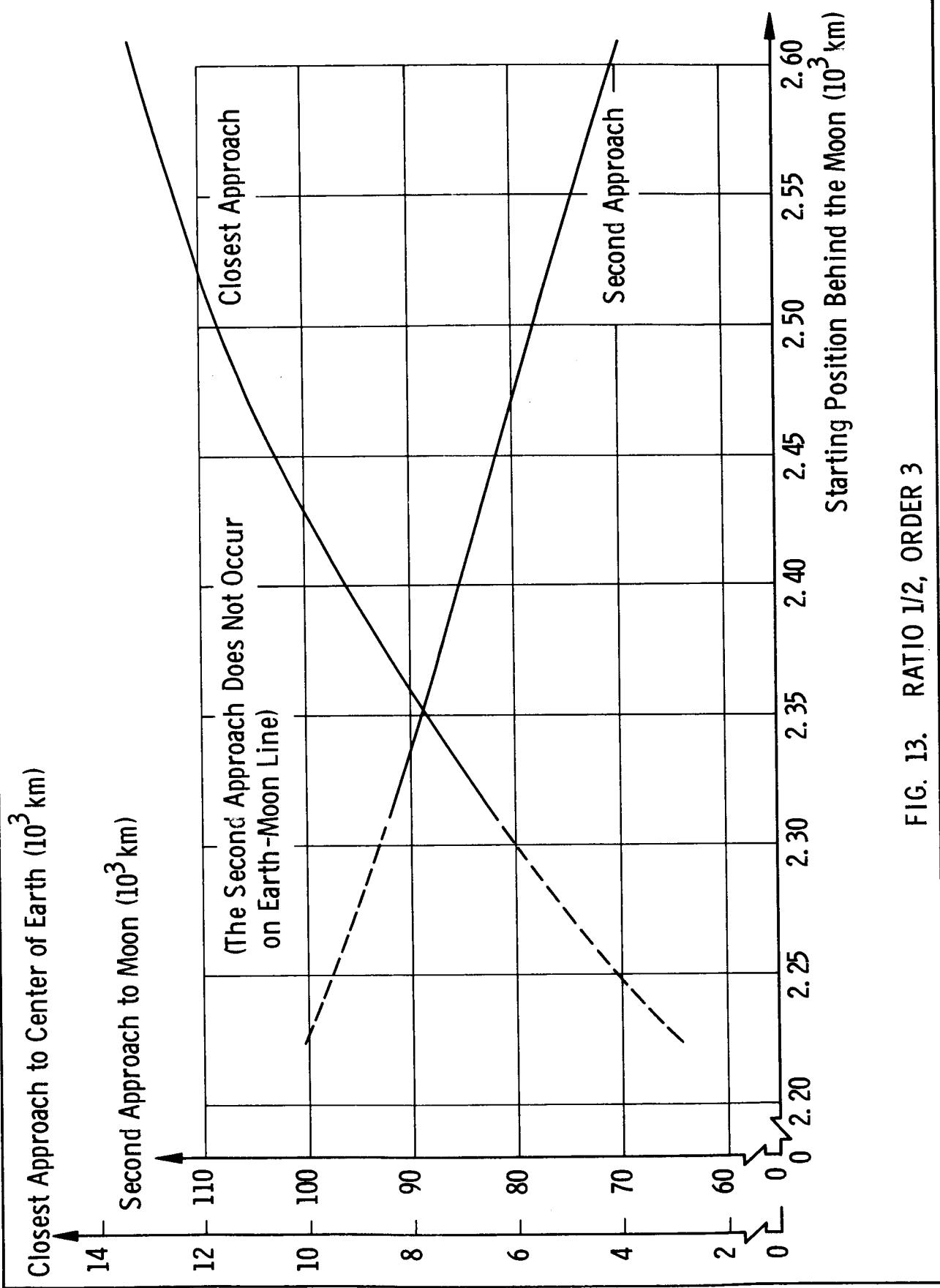


FIG. 13. RATIO 1/2, ORDER 3

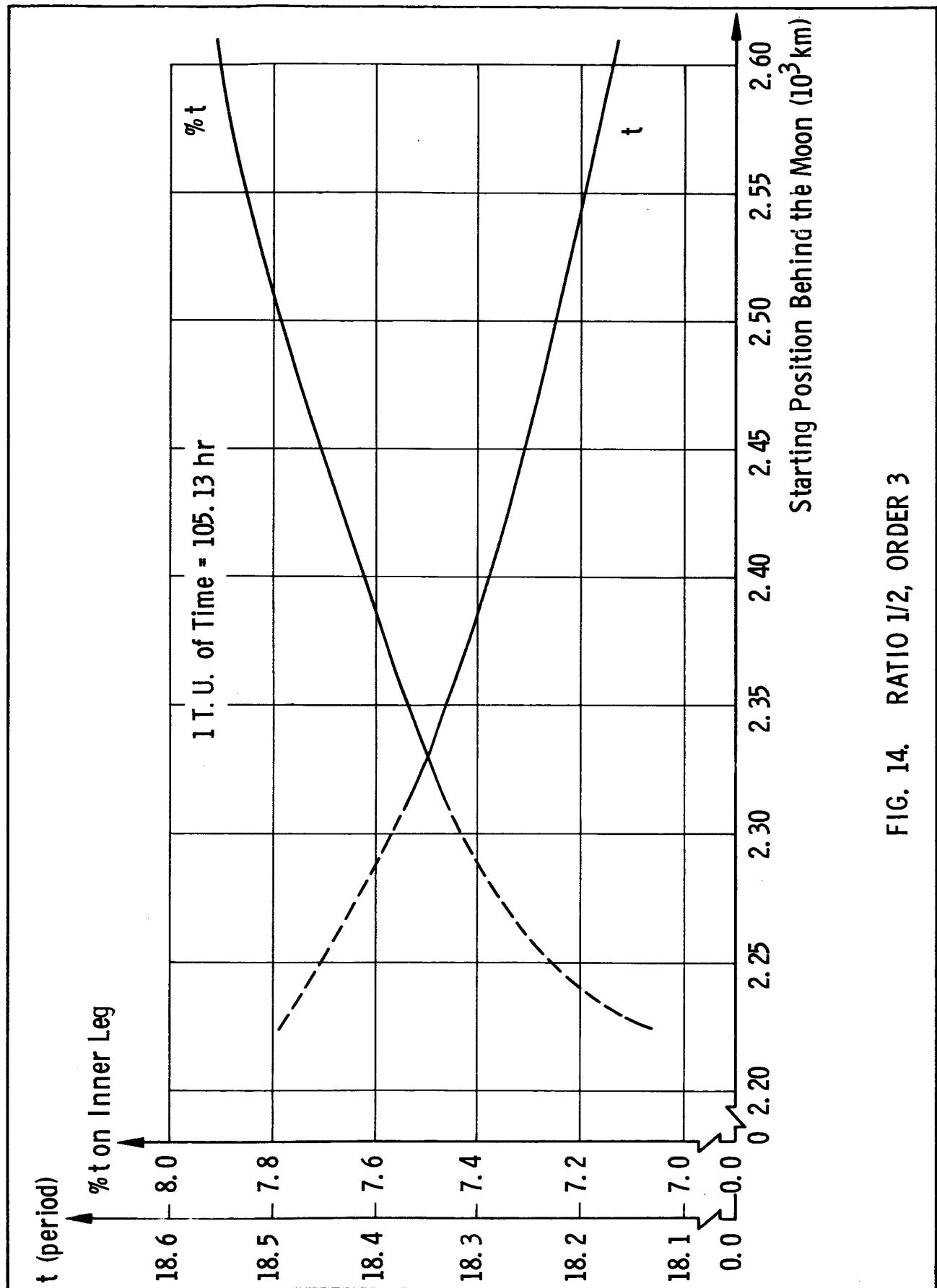


FIG. 14. RATIO 1/2, ORDER 3

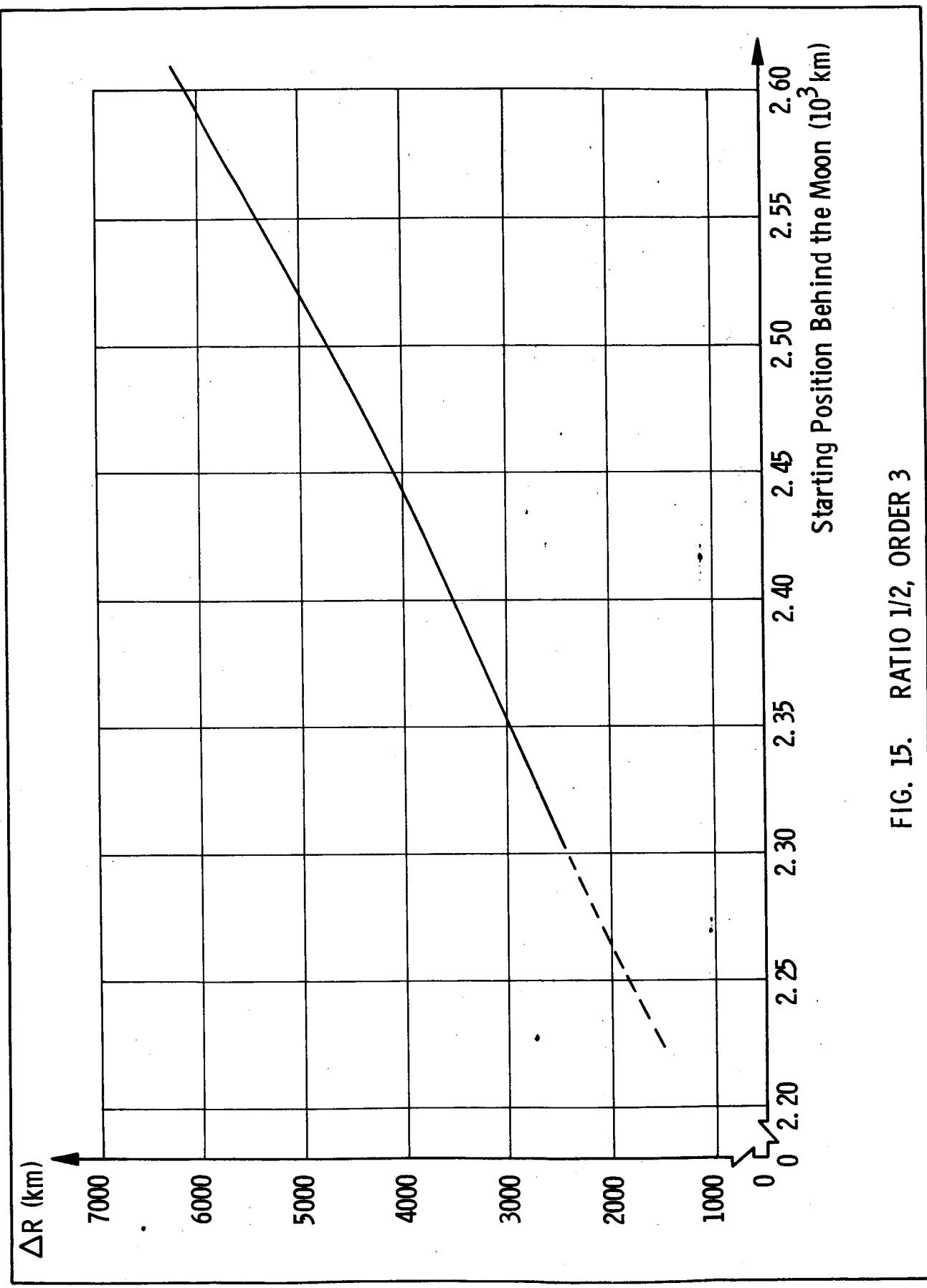


FIG. 15. RATIO 1/2, ORDER 3

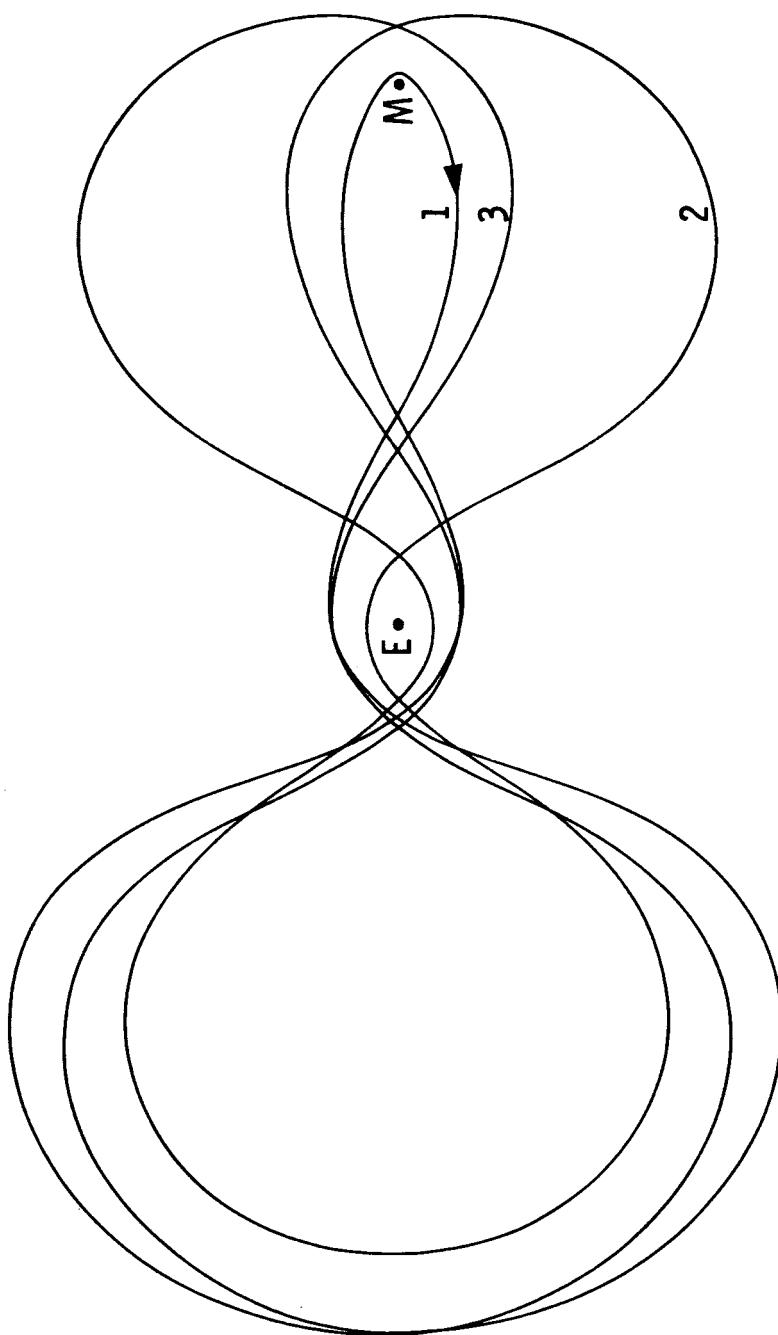


FIG. 16. RATIO 1/2, ORDER 3

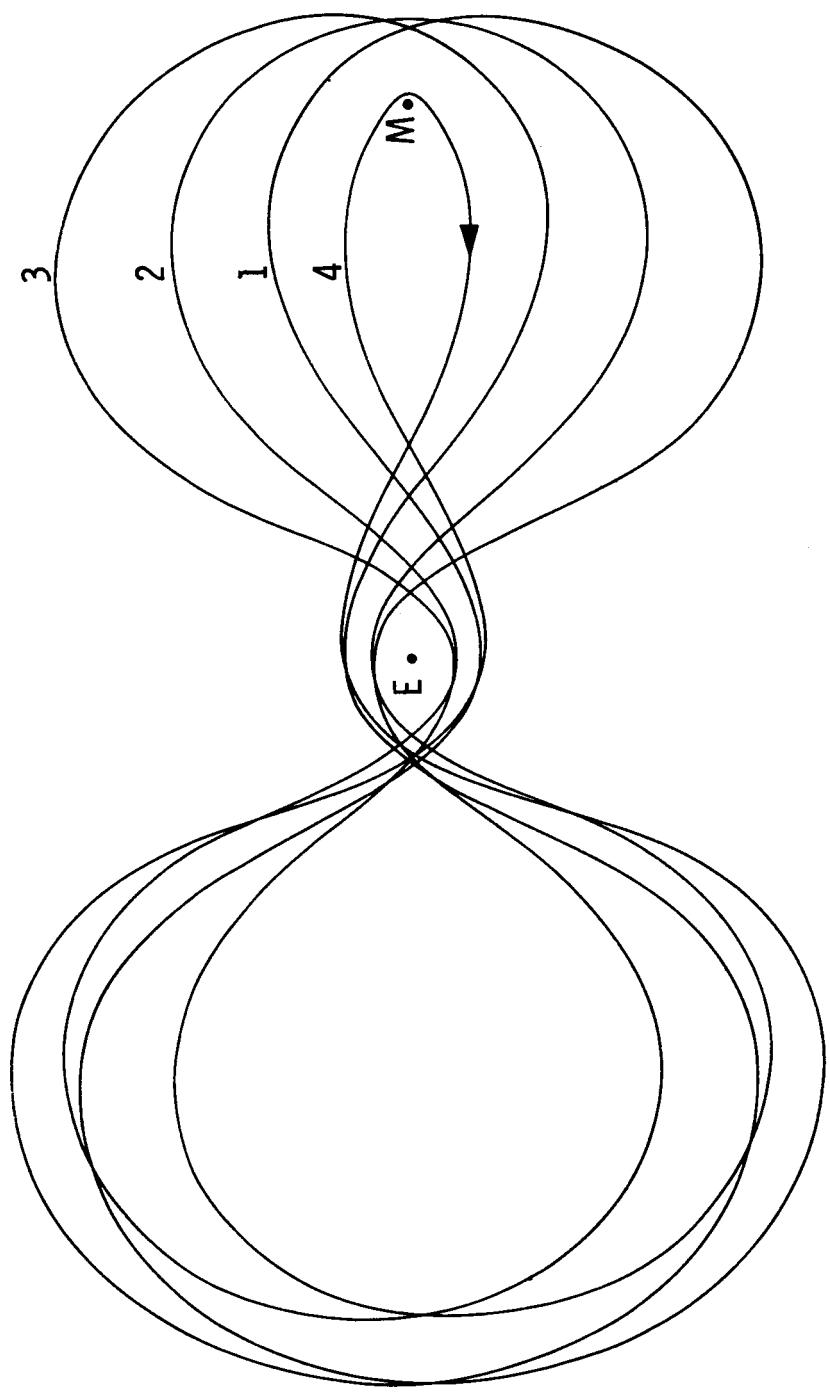


FIG. 17. RATIO 1/2, ORDER 4, CLASS A

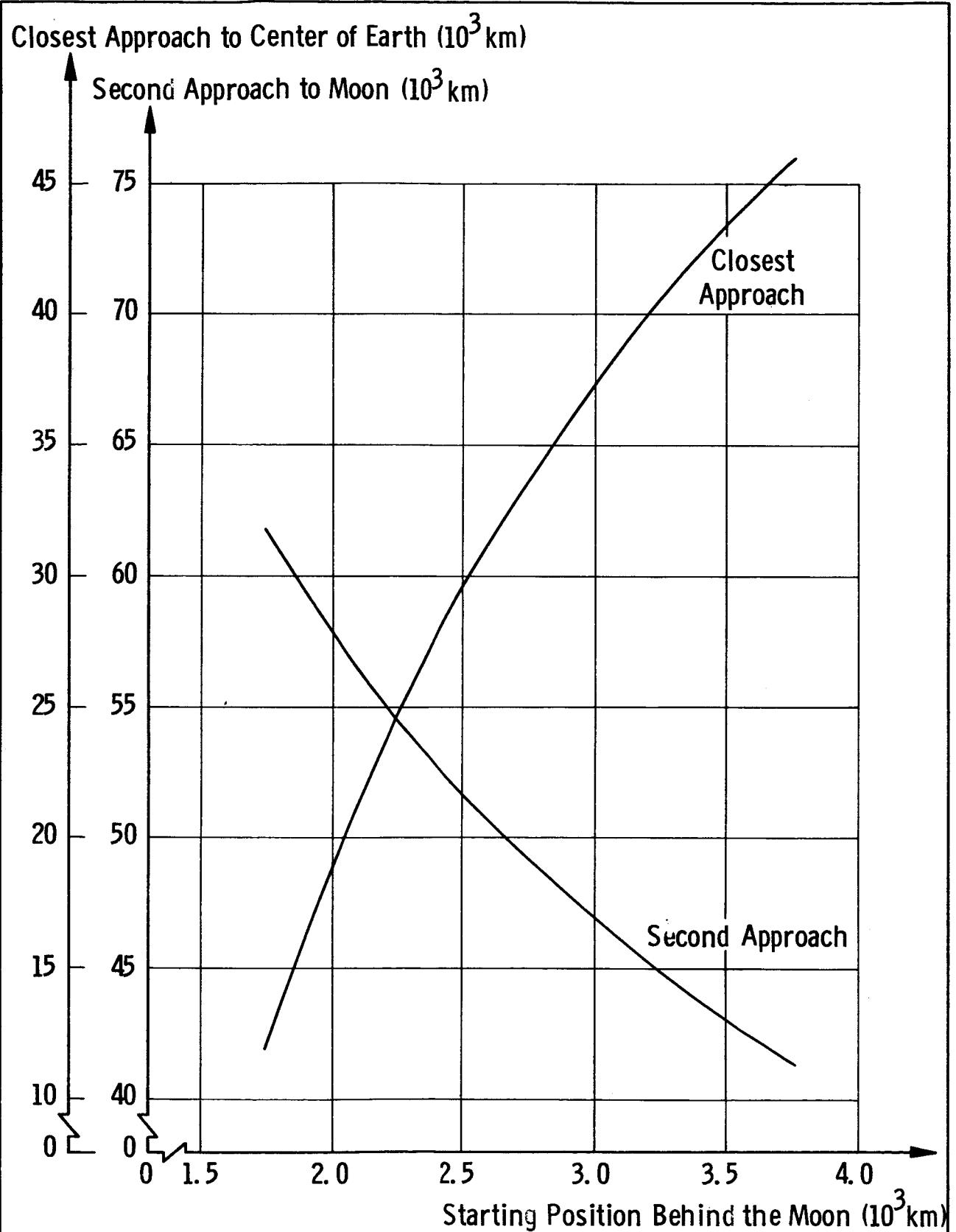


FIG. 18. RATIO 1/2, ORDER 4, CLASS A

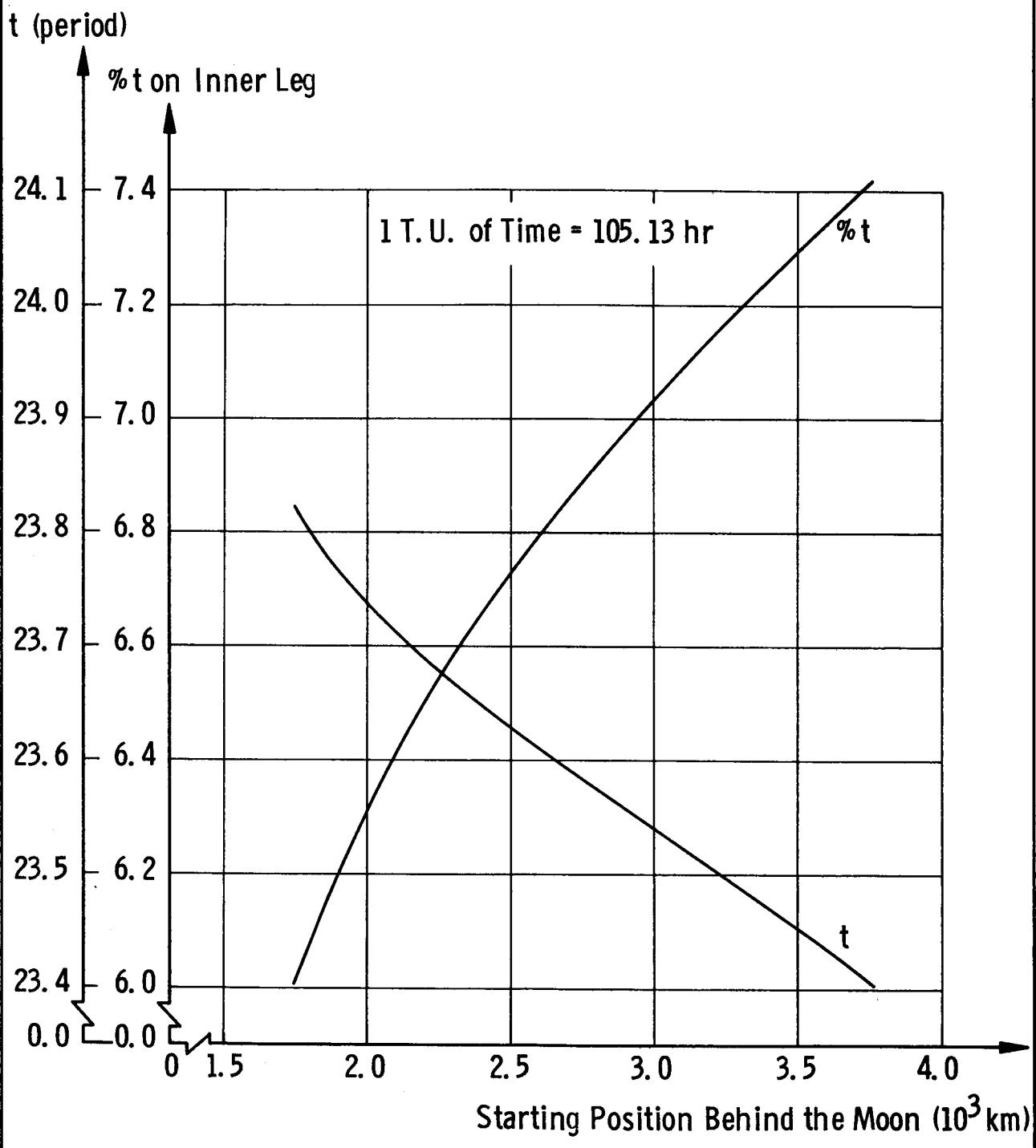
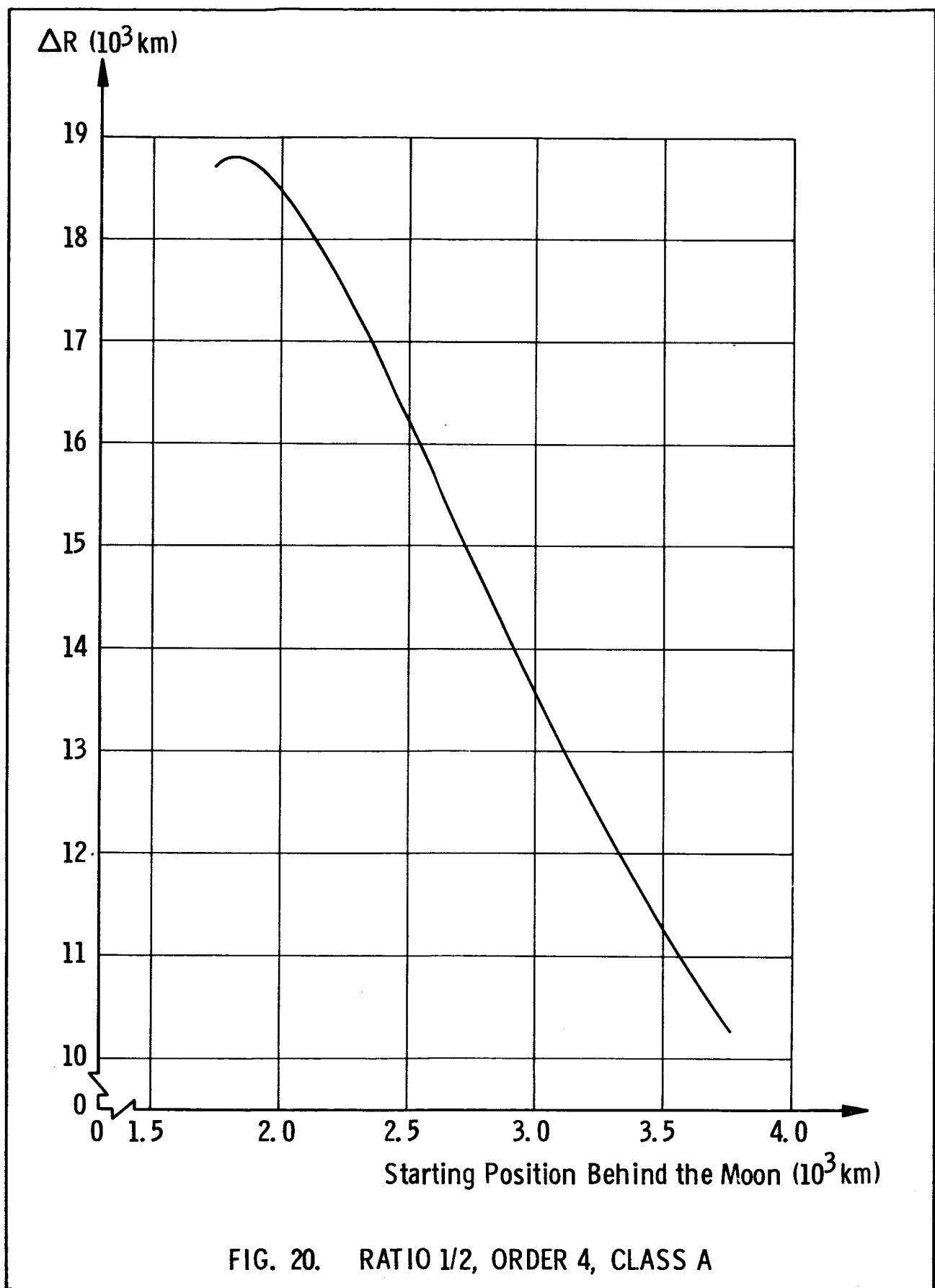


FIG. 19. RATIO 1/2, ORDER 4, CLASS A



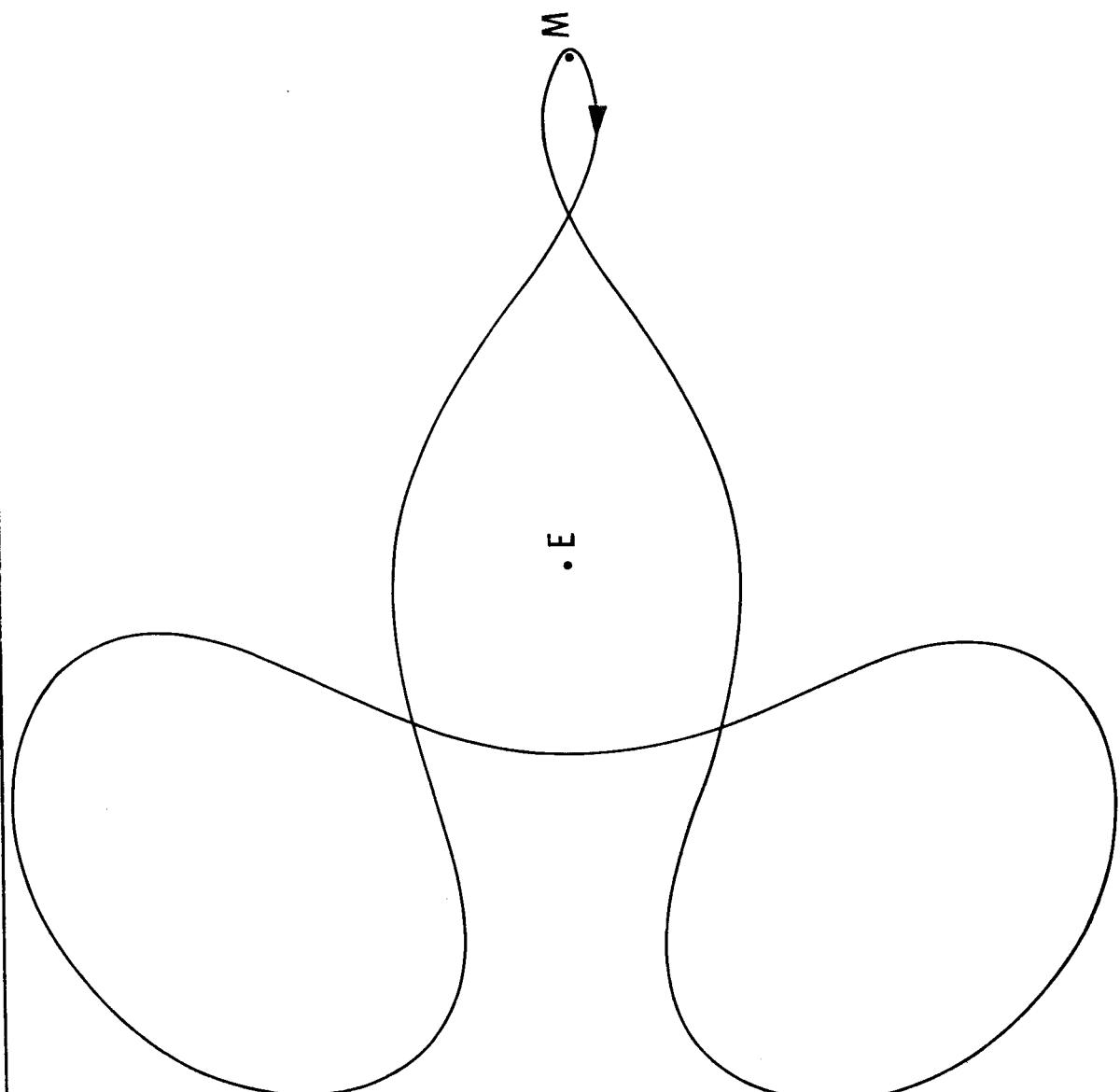


FIG. 21. RATIO 2/3, ORDER 1

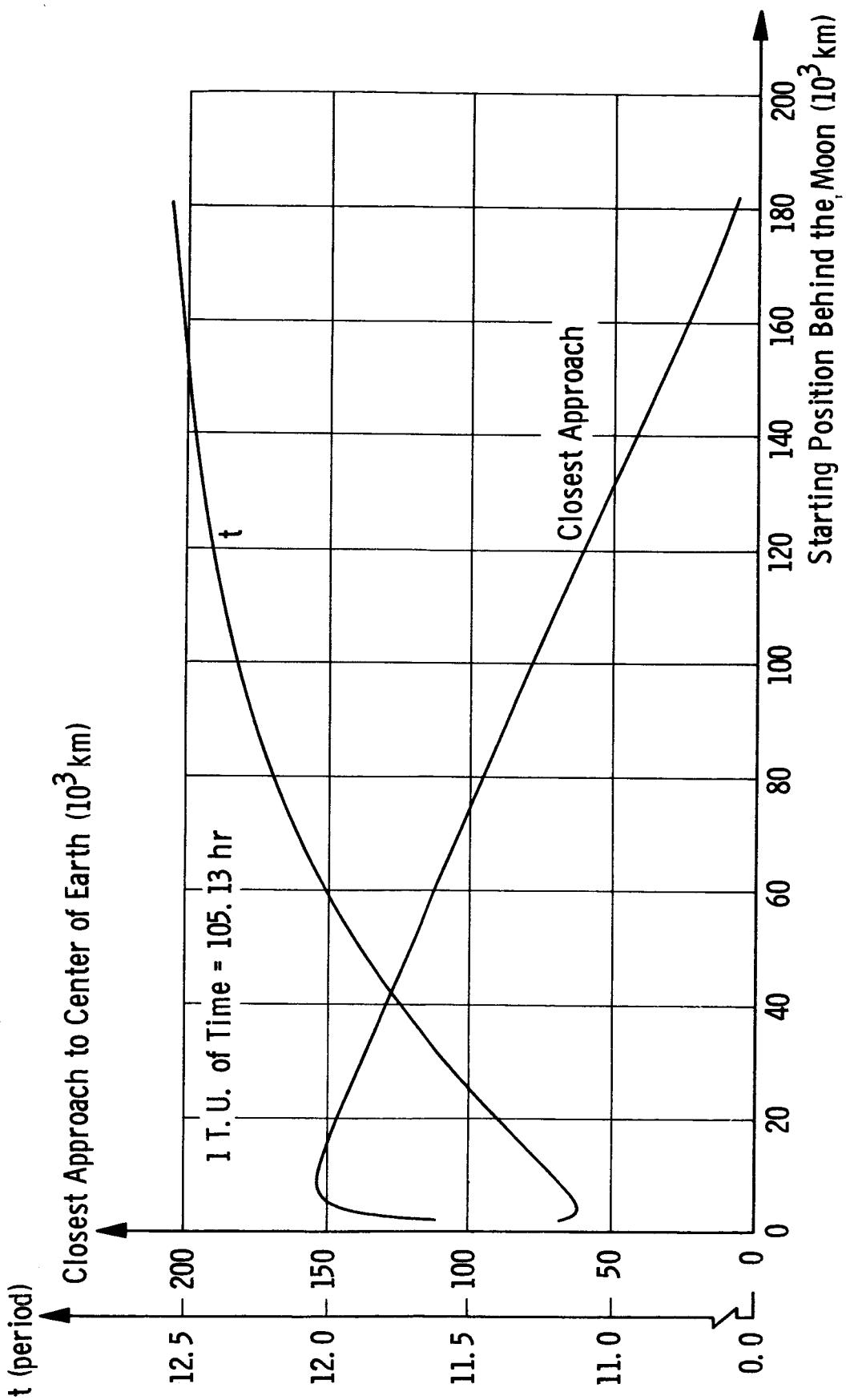


FIG. 22. RATIO 2/3, ORDER 1

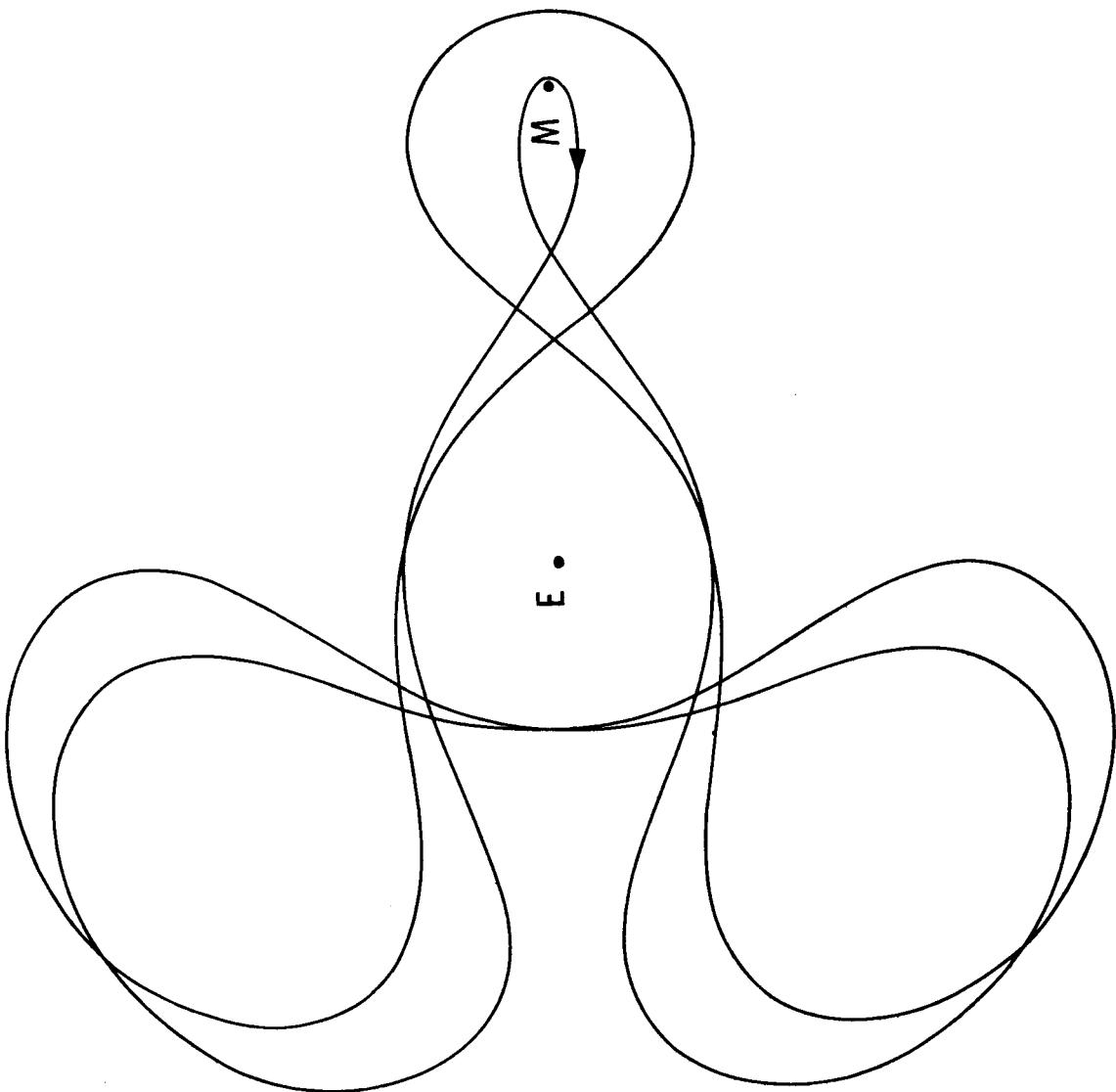


FIG. 23. RATIO 2/3, ORDER 2, CLASS A

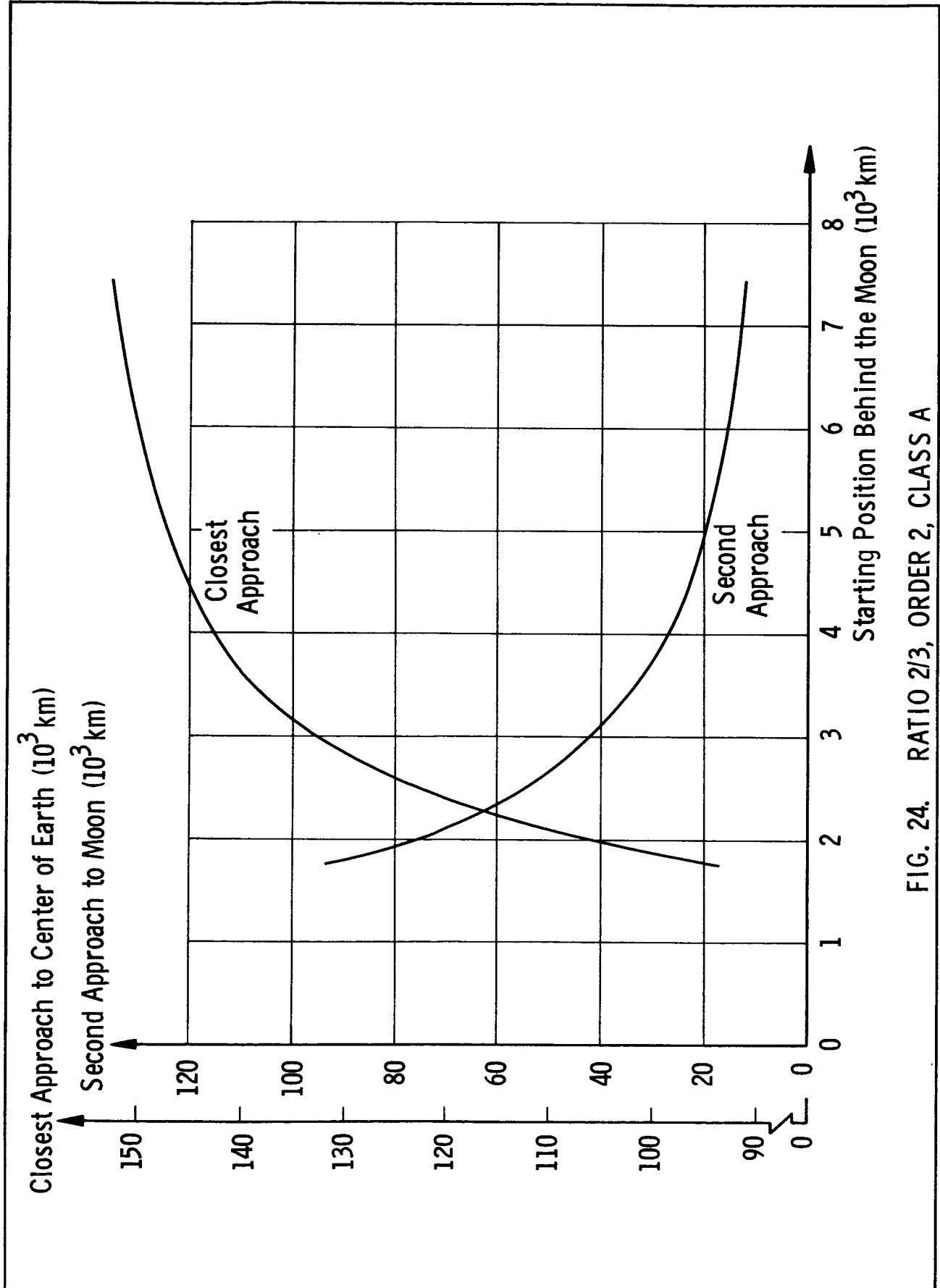


FIG. 24. RATIO 2/3, ORDER 2, CLASS A

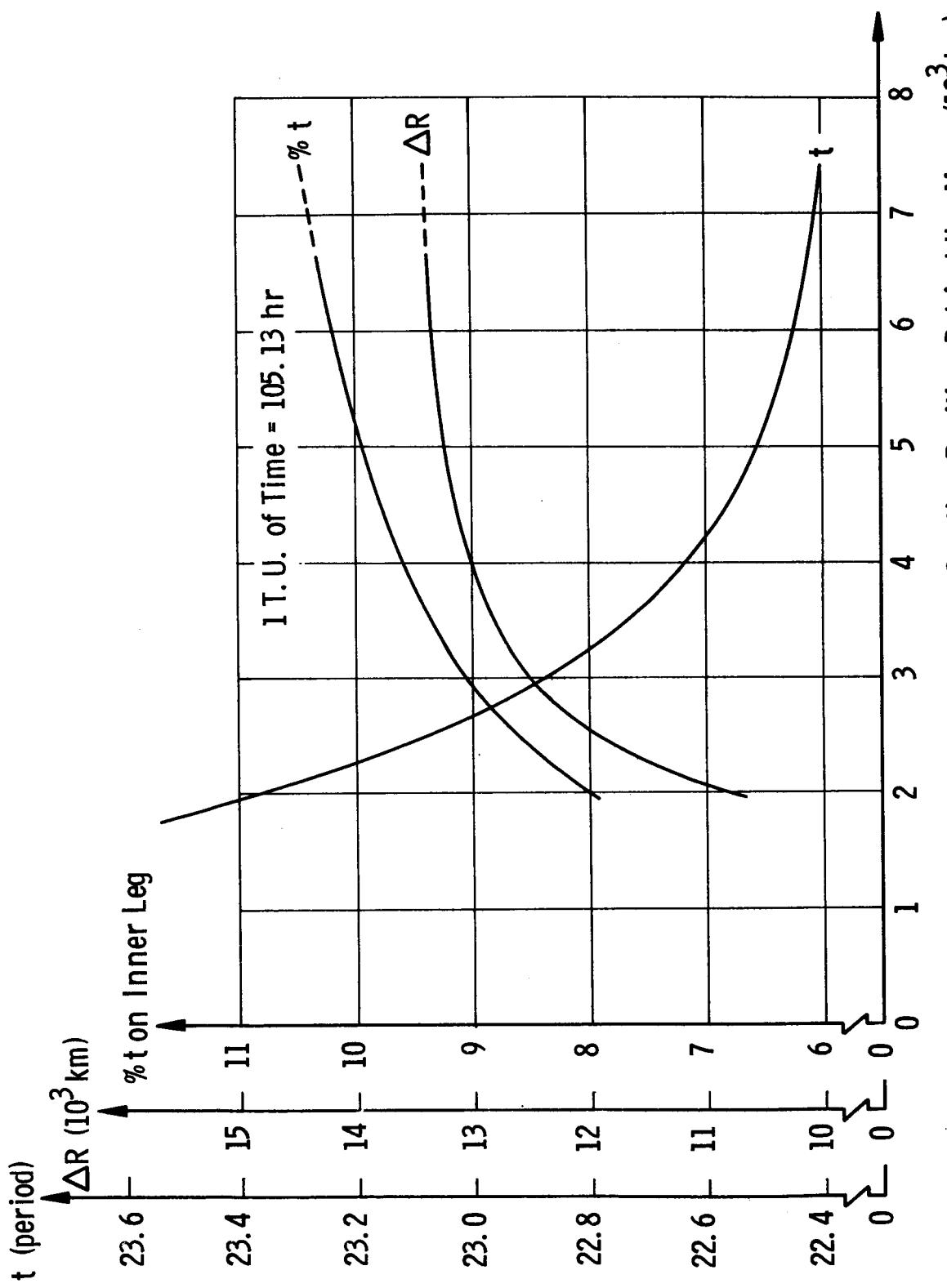


FIG. 25. RATIO 2/3, ORDER 2, CLASS A

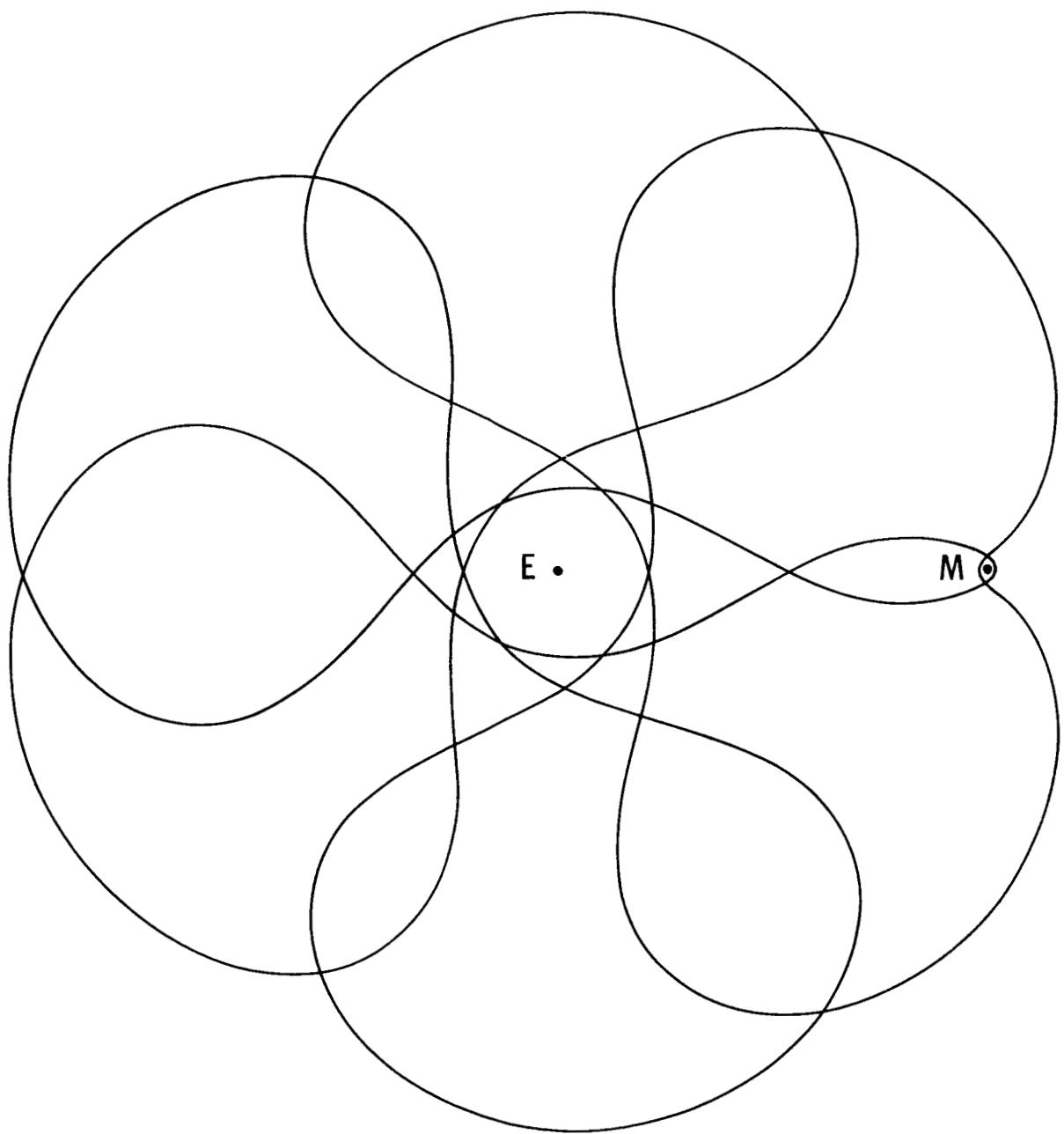


FIG. 26. RATIO 2/3, ORDER 2, CLASS B

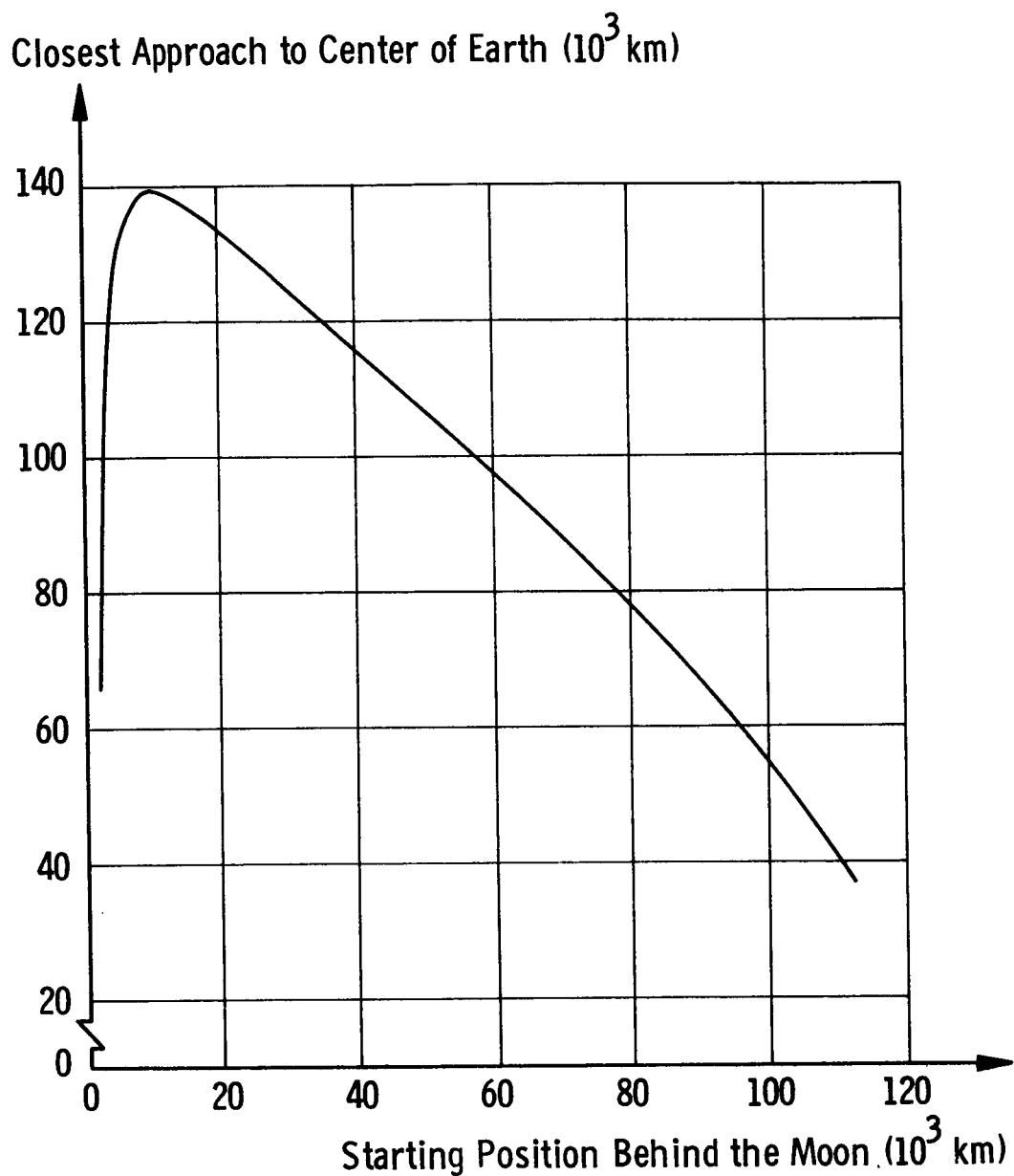


FIG. 27. RATIO 2/3, ORDER 2, CLASS B

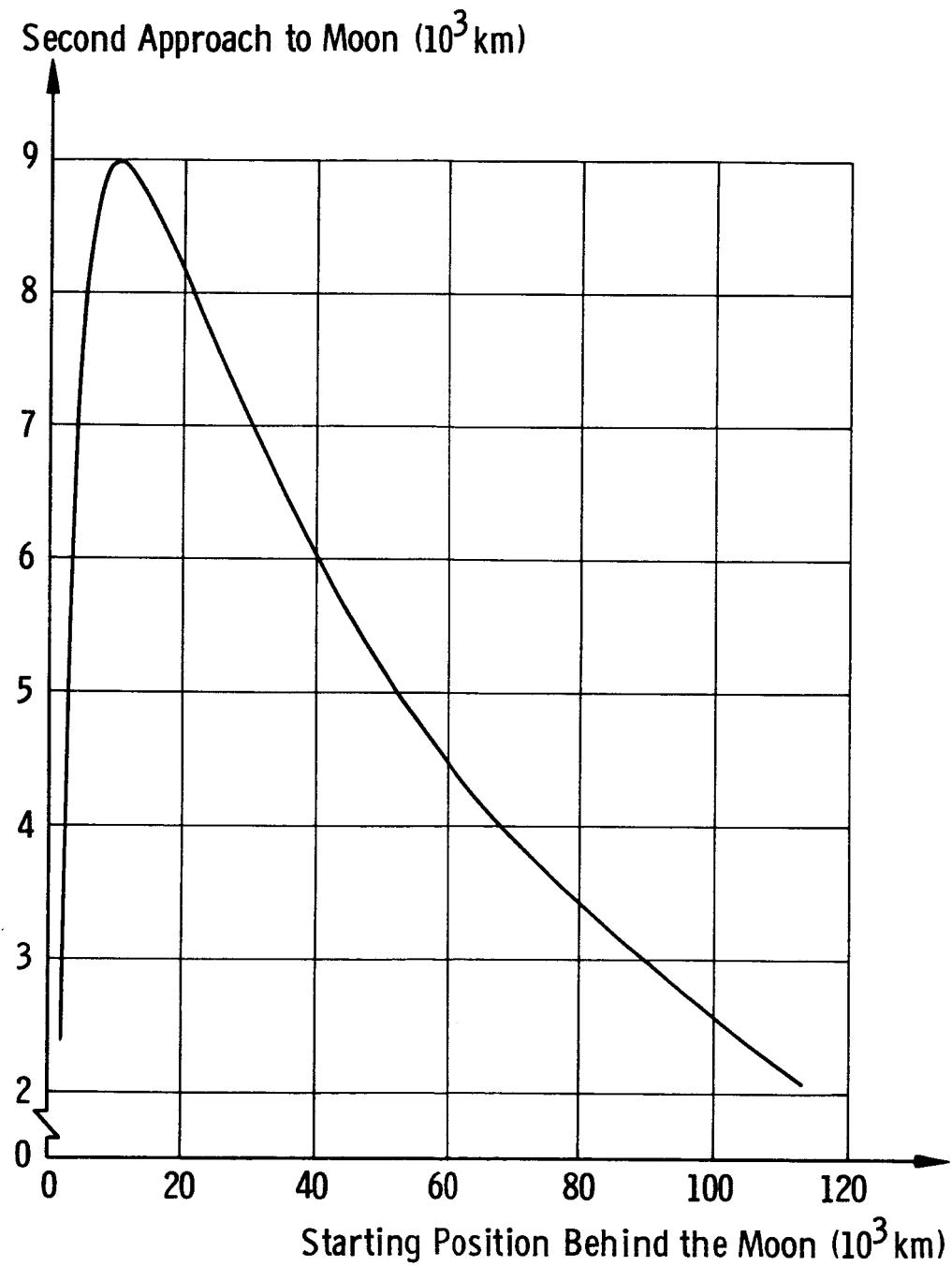


FIG. 28. RATIO 2/3, ORDER 2, CLASS B

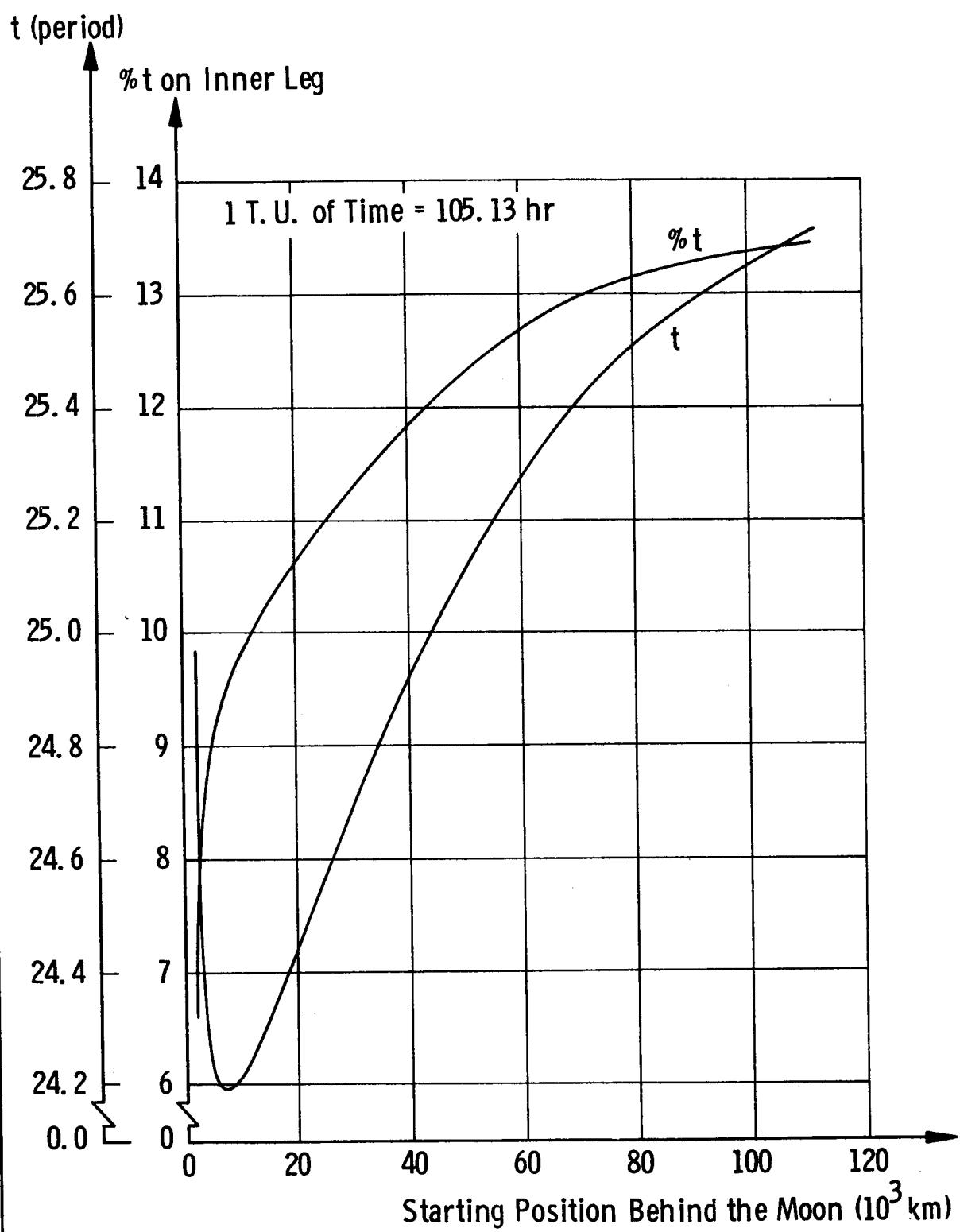


FIG. 29. RATIO 2/3, ORDER 2, CLASS B

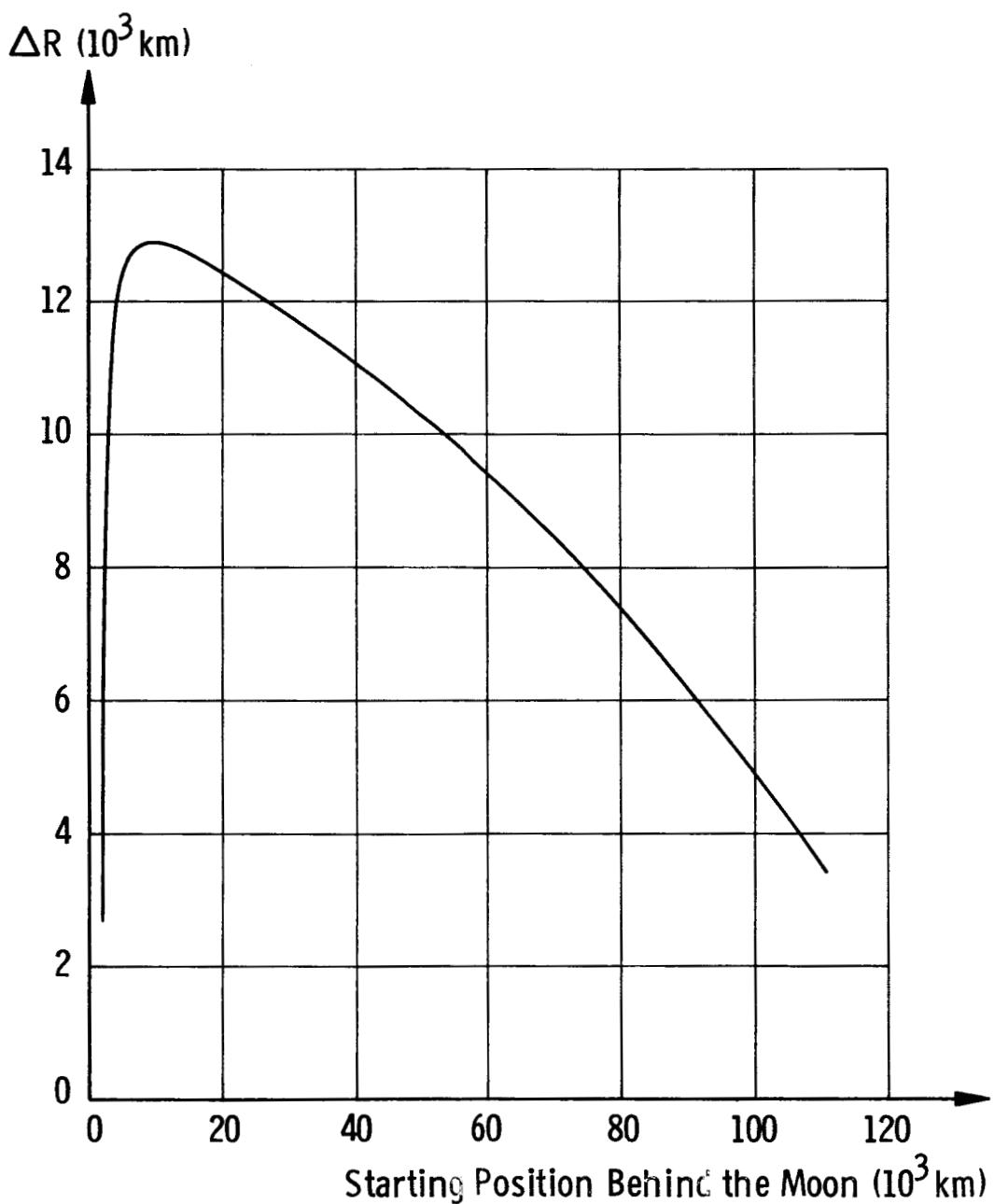


FIG. 30. RATIO 2/3, ORDER 2, CLASS B

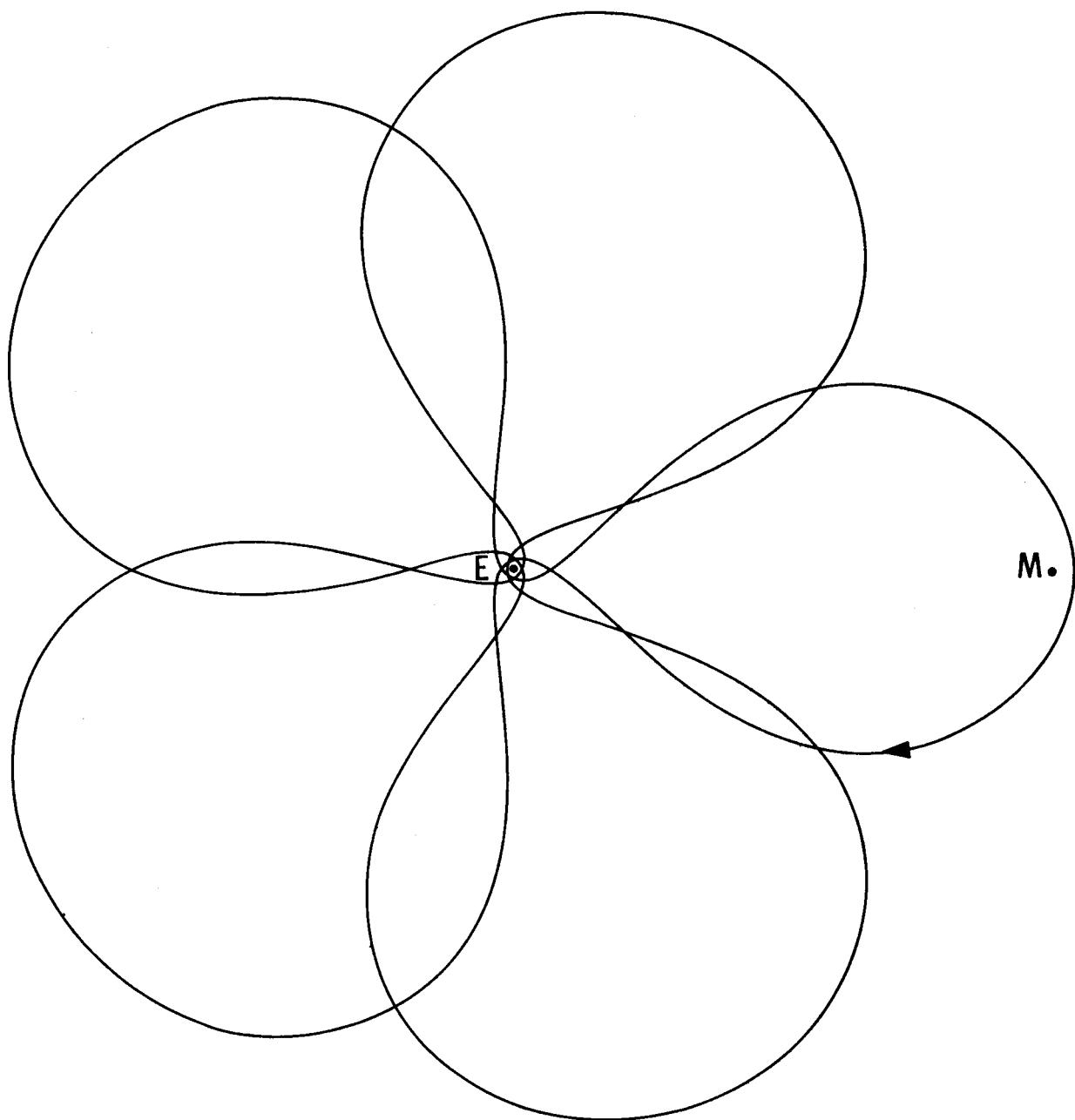


FIG. 31. RATIO 2/5, ORDER 1

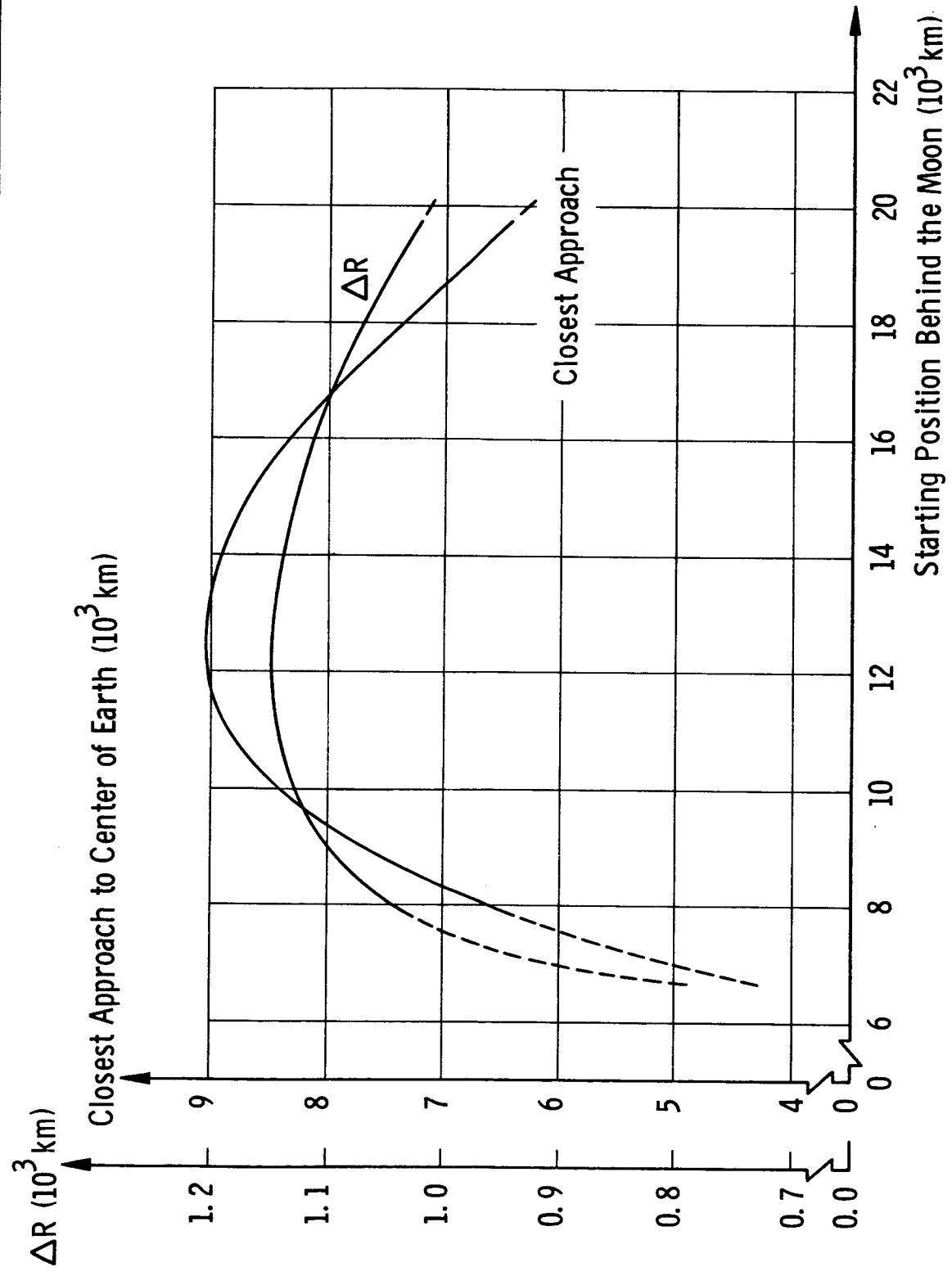


FIG. 32. RATIO 2/5, ORDER 1

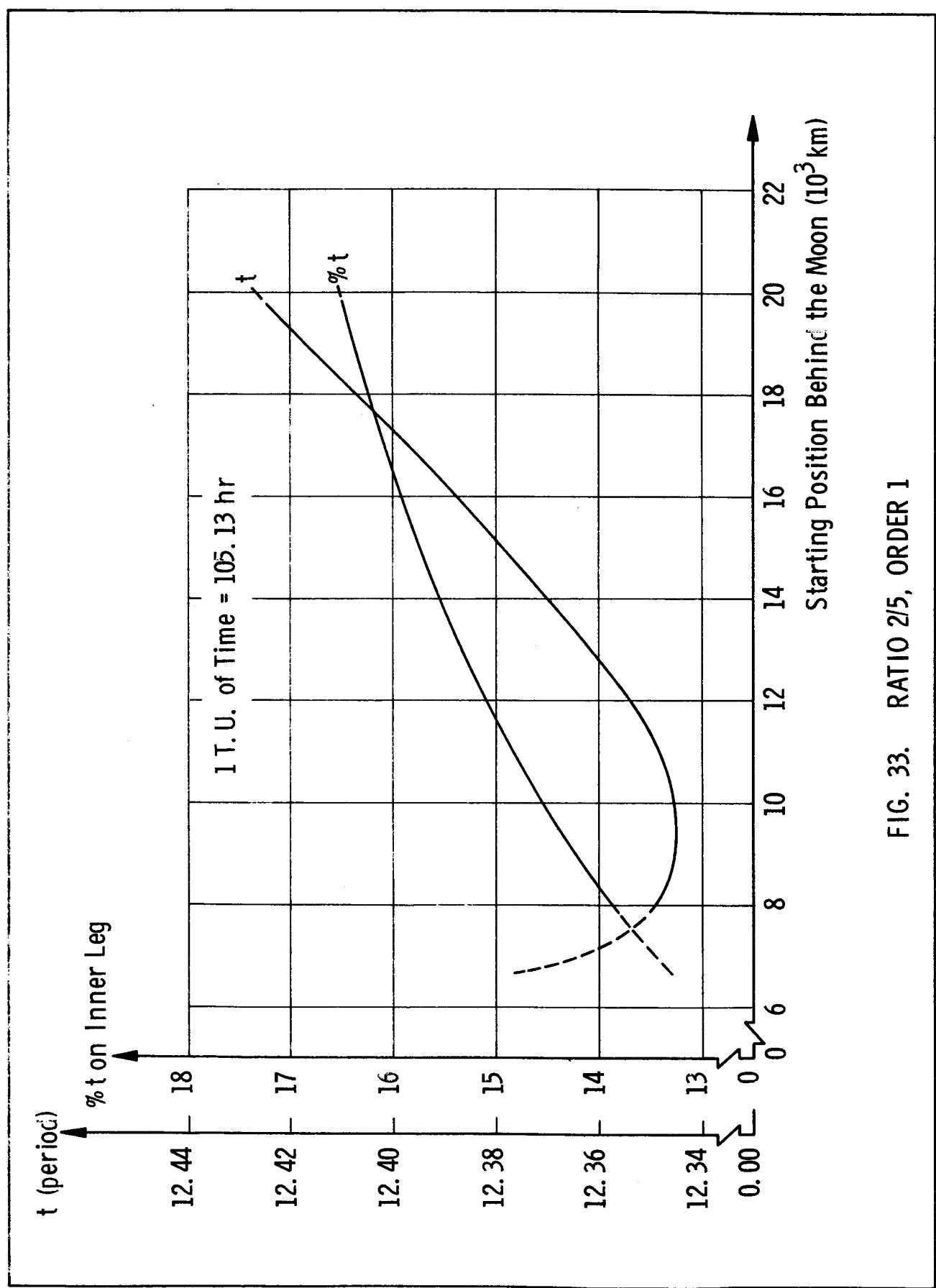


FIG. 33. RATIO 2/5, ORDER 1

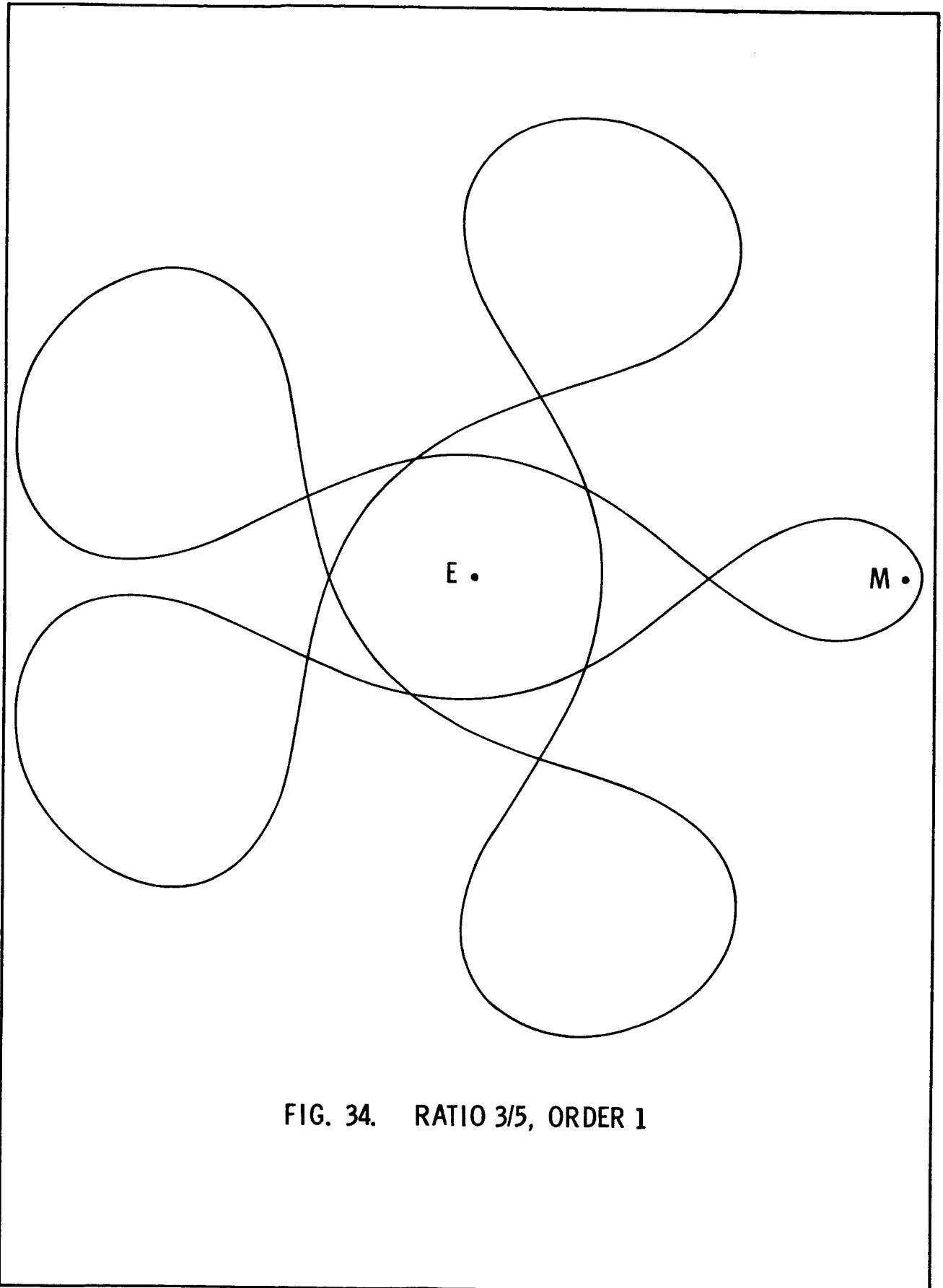


FIG. 34. RATIO 3/5, ORDER 1

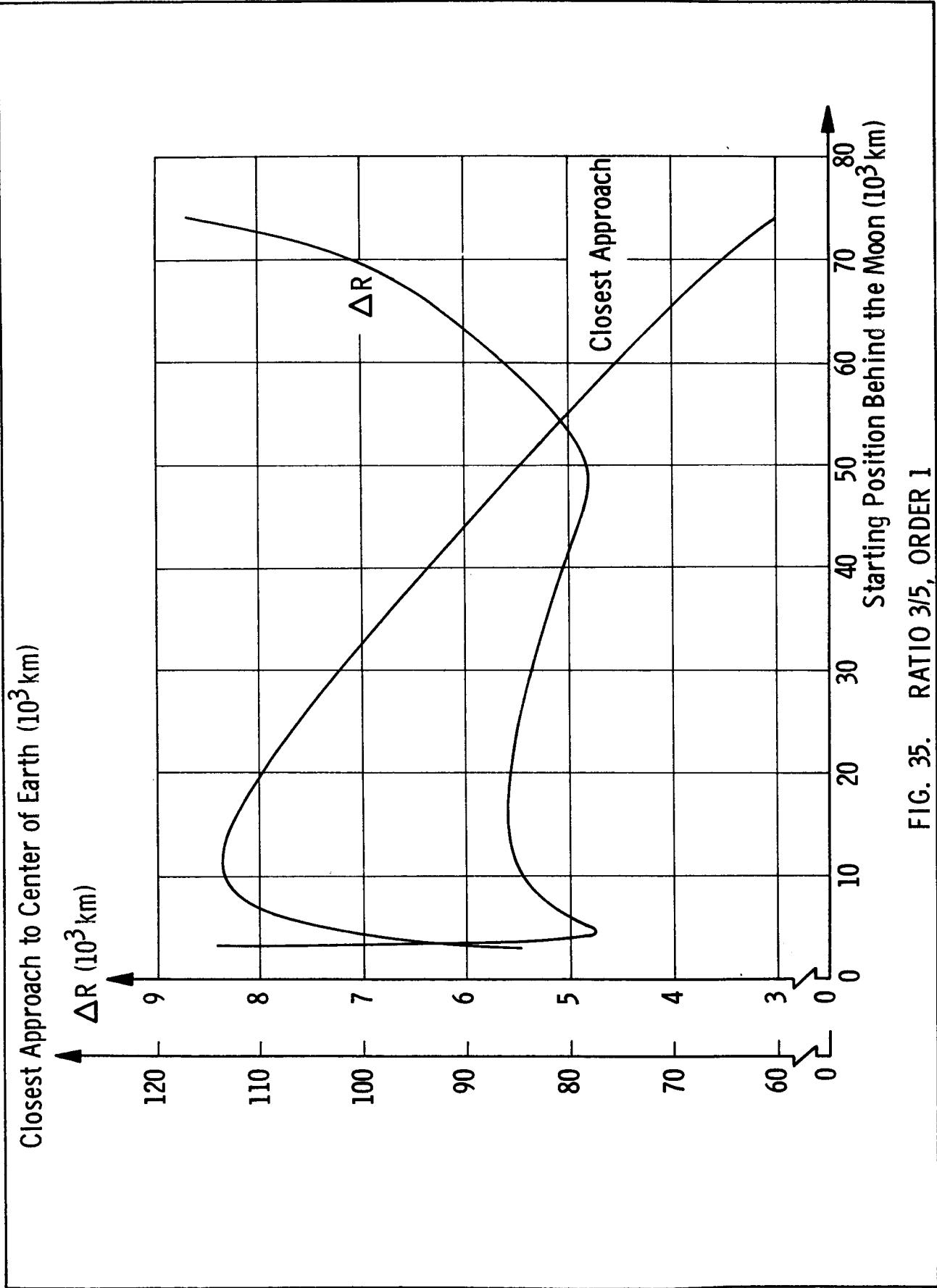


FIG. 35. RATIO 3/5, ORDER 1

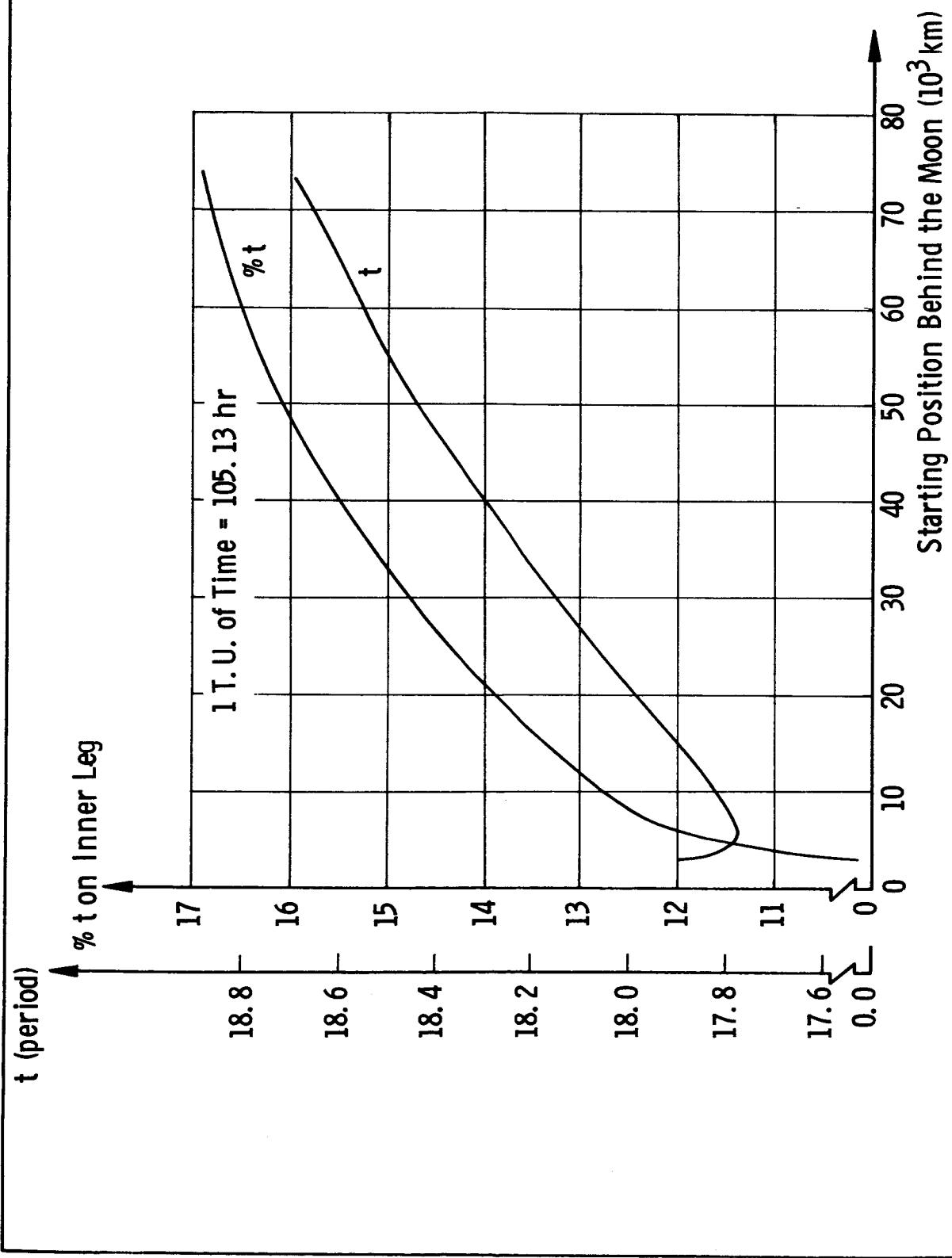


FIG. 36. RATIO 3/5, ORDER 1

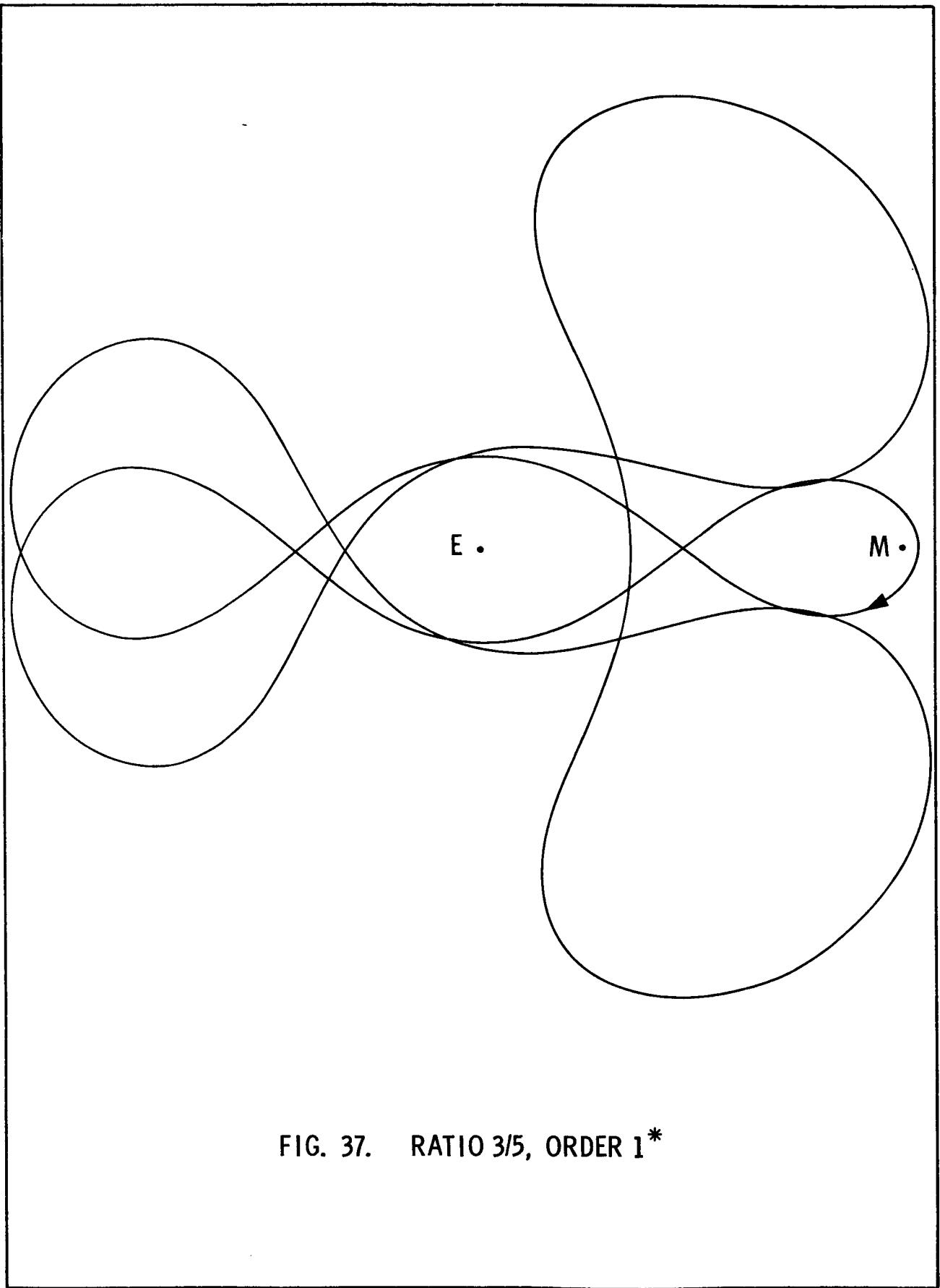


FIG. 37. RATIO 3/5, ORDER 1\*

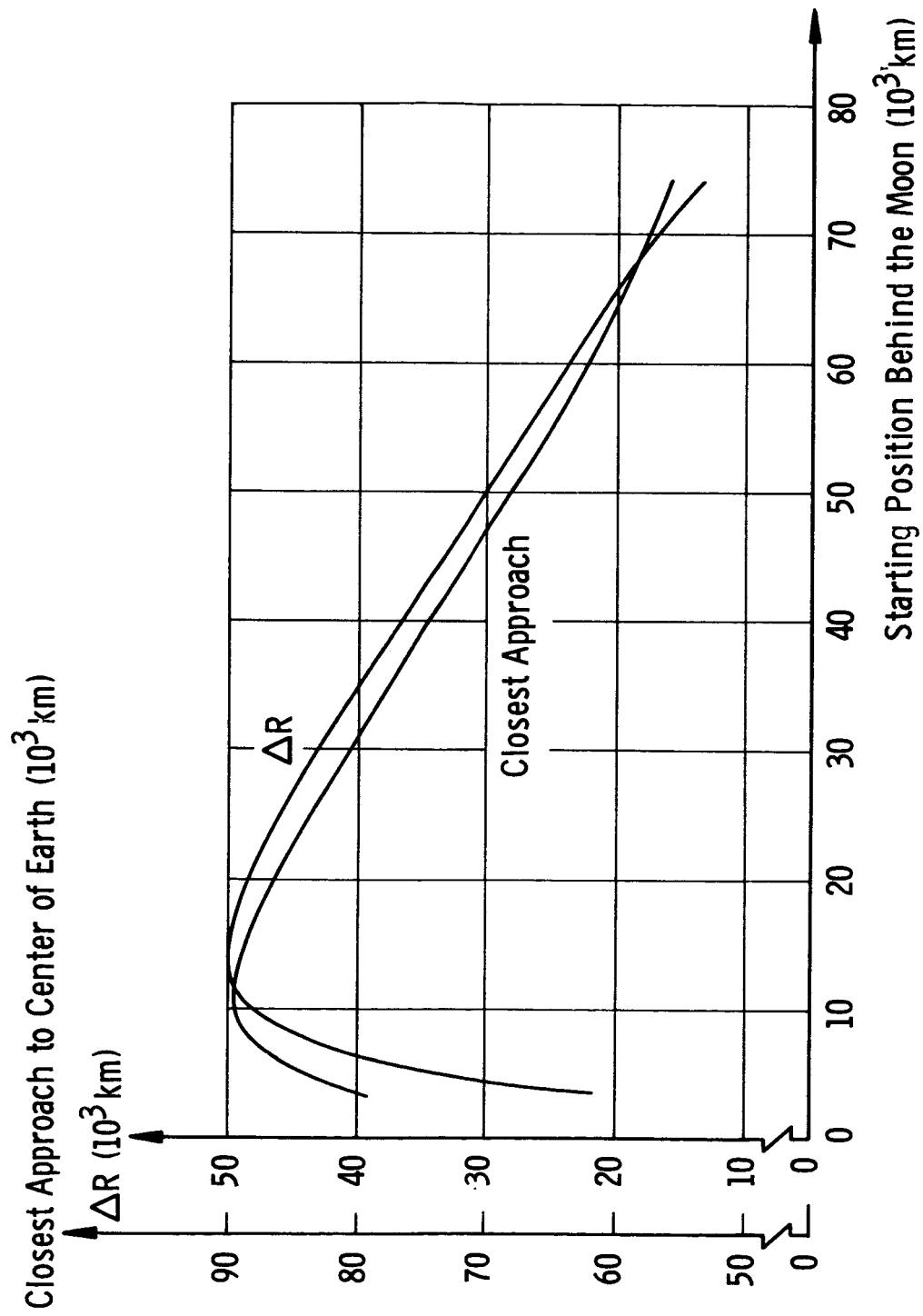


FIG. 38. RATIO 3/5, ORDER 1 \*

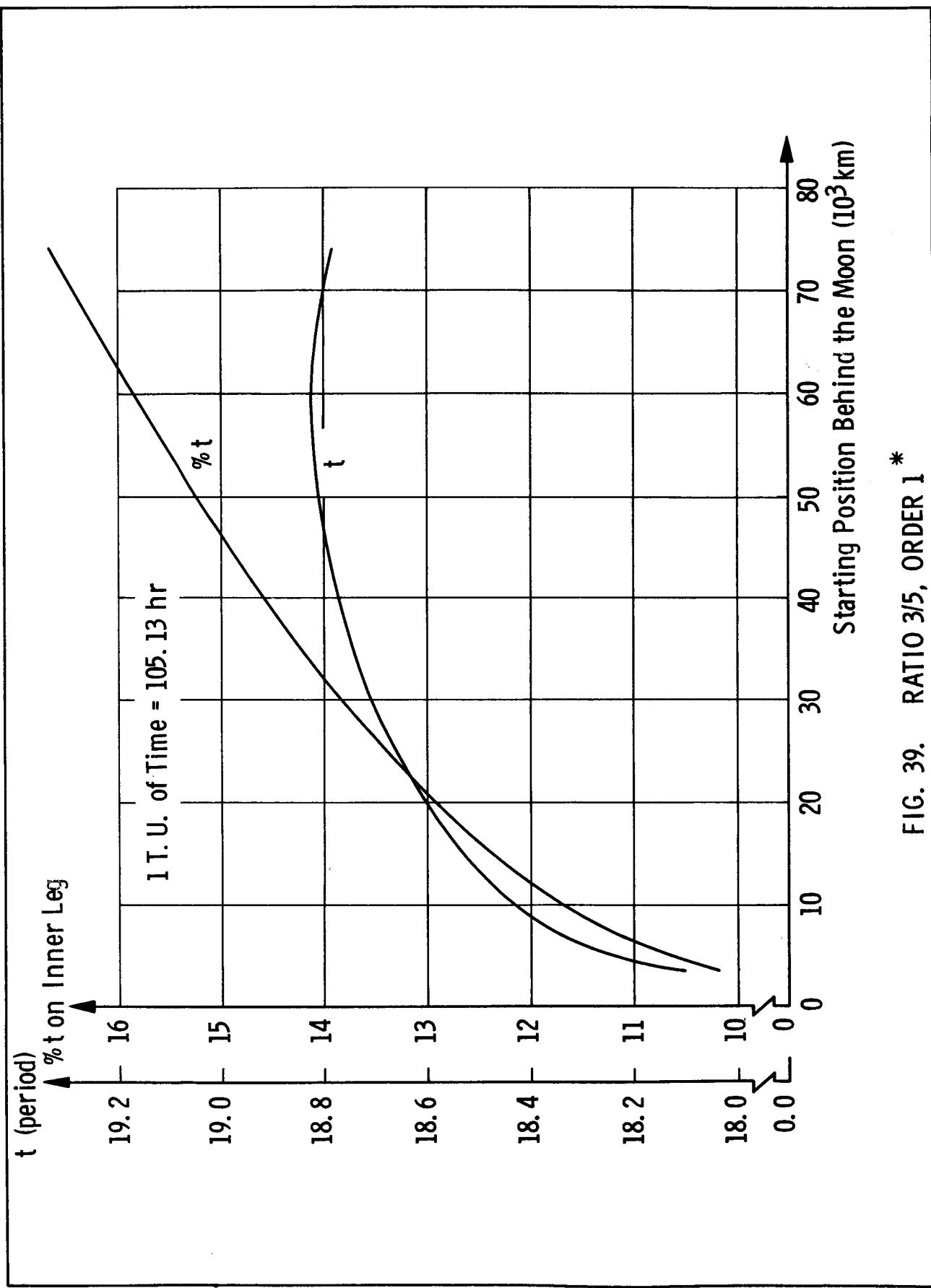


FIG. 39. RATIO 3/5, ORDER 1 \*

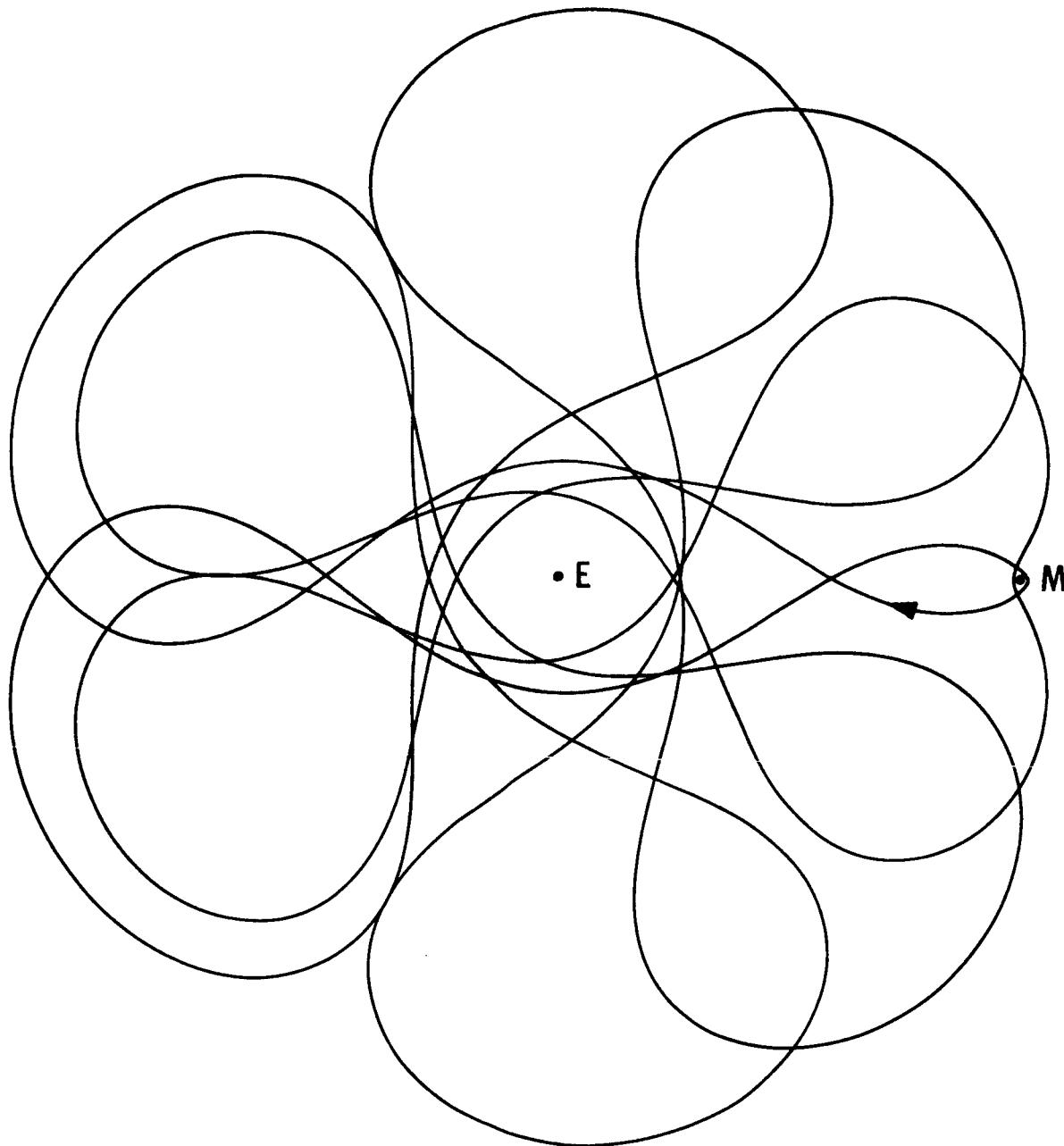


FIG. 40. RATIO 3/5, ORDER 2, CLASS B

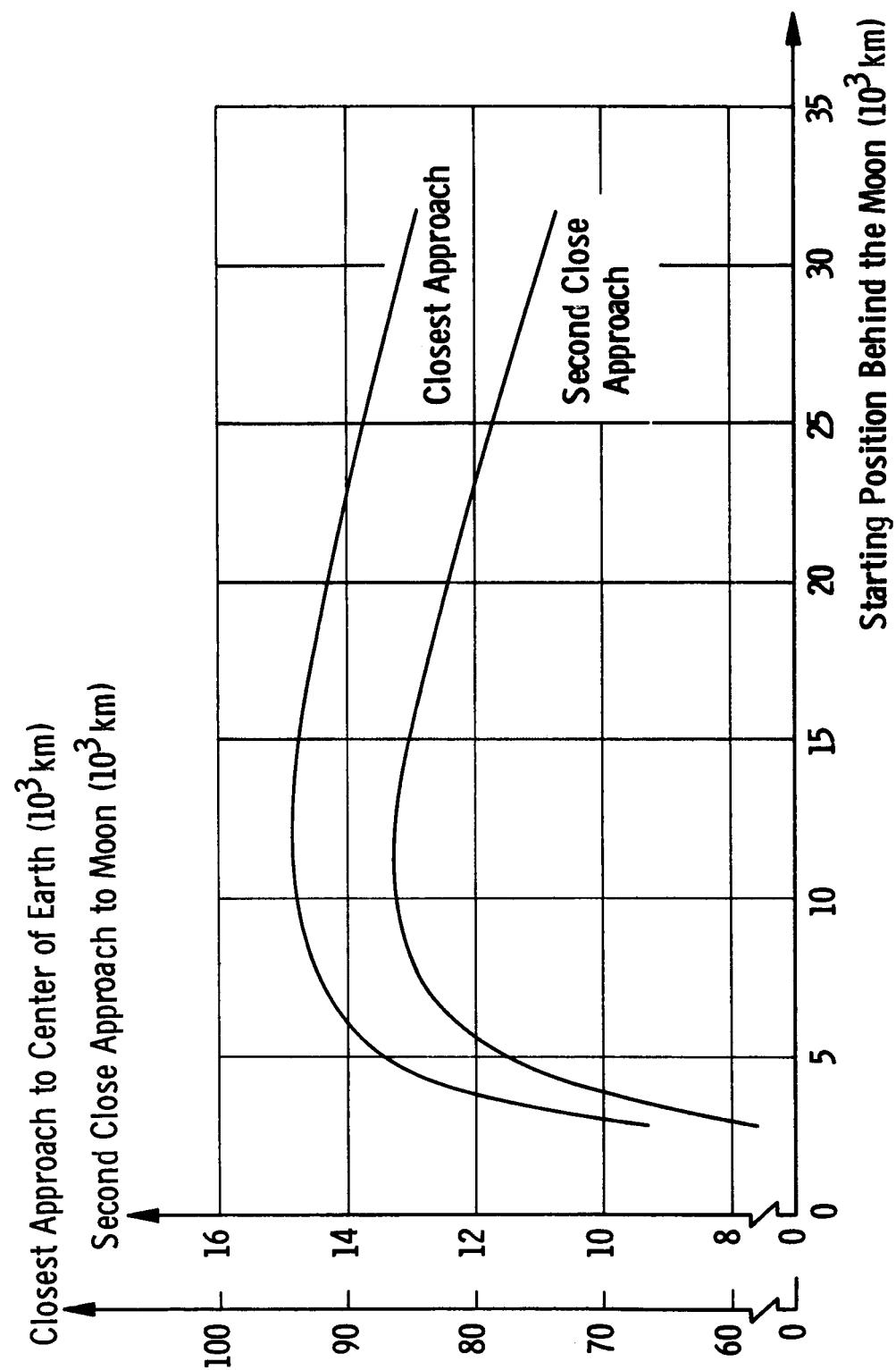


FIG. 41. RATIO 3/5, ORDER 2, CLASS B

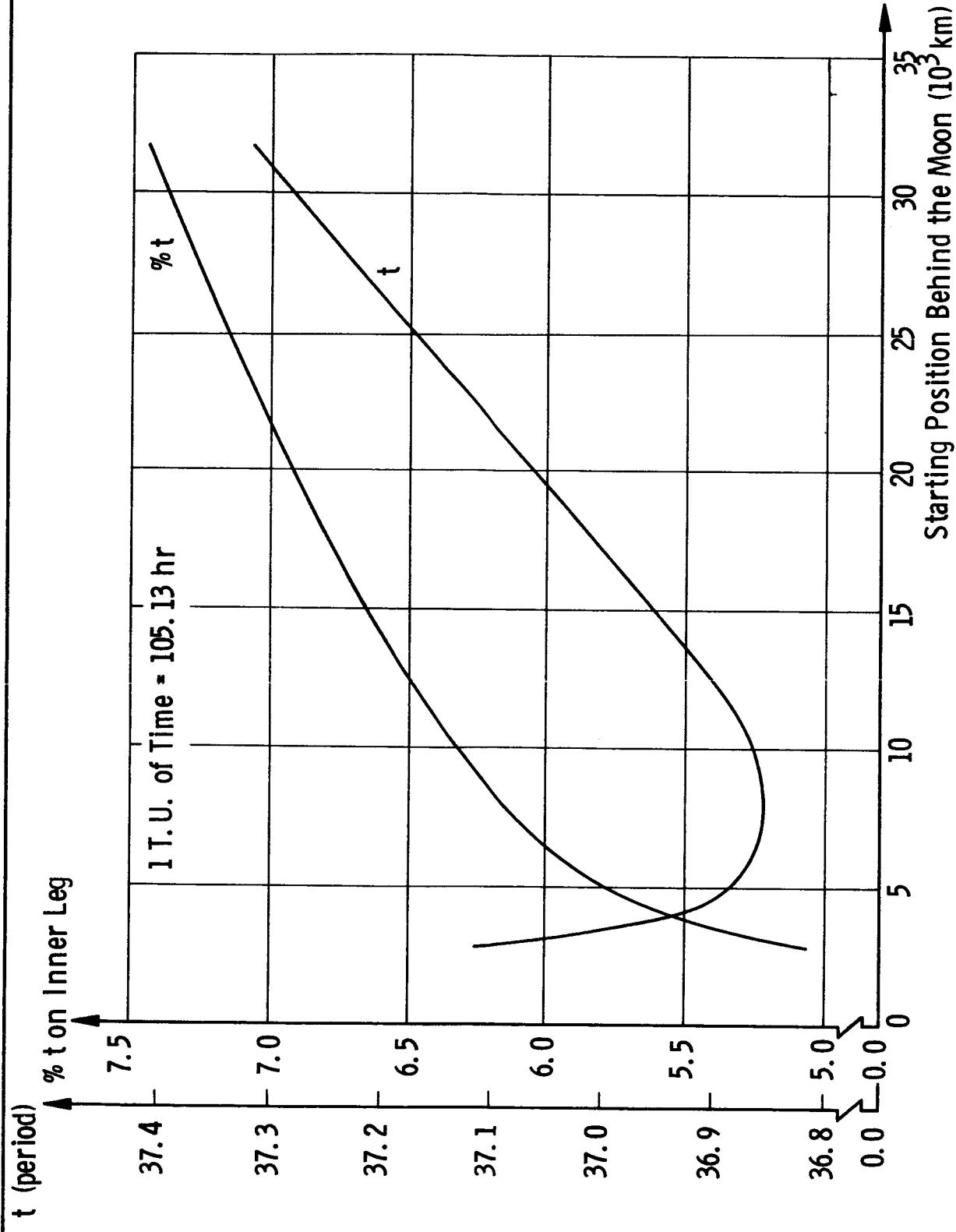


FIG. 42. RATIO 3/5, ORDER 2, CLASS B

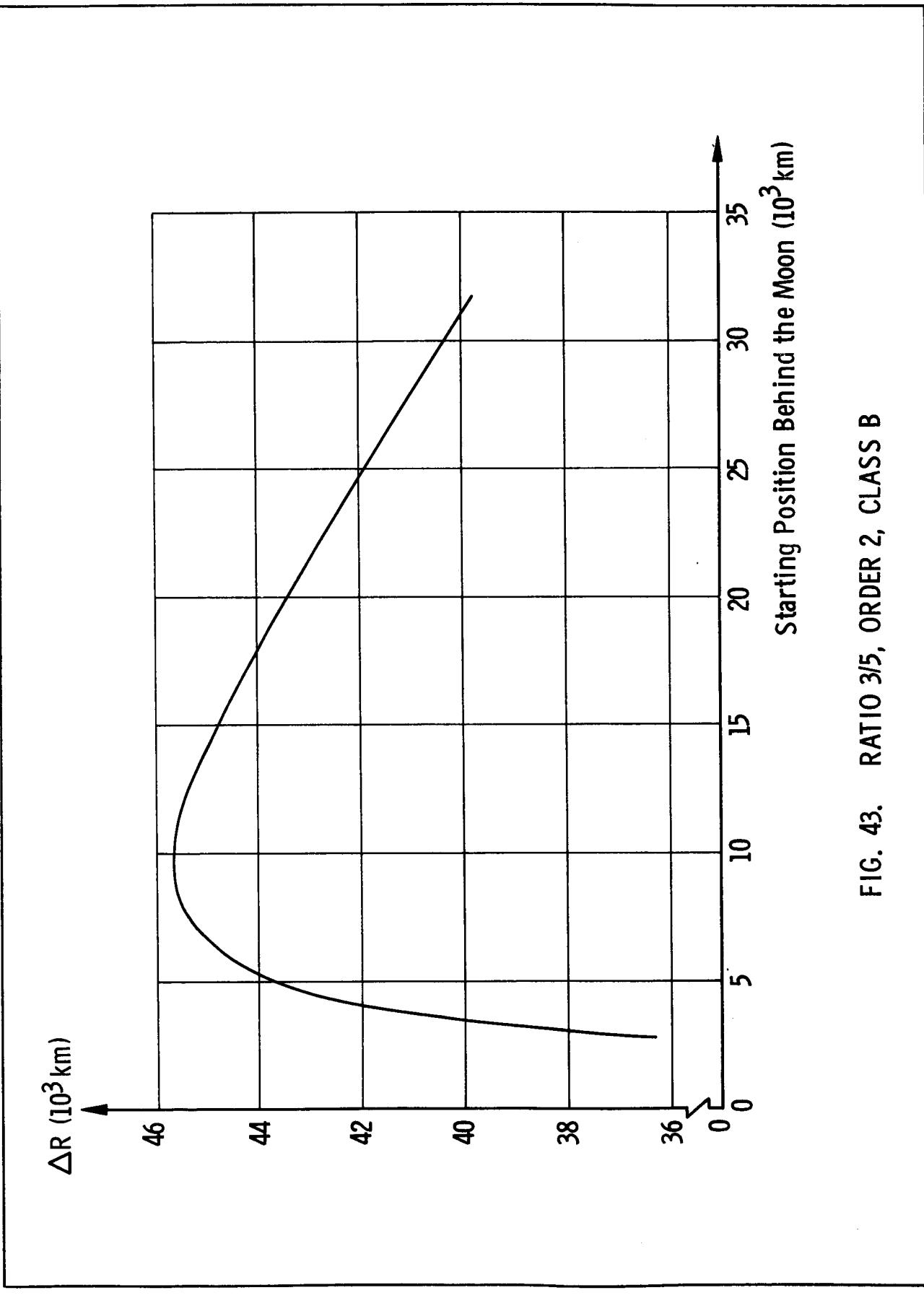


FIG. 43. RATIO 3/5, ORDER 2, CLASS B

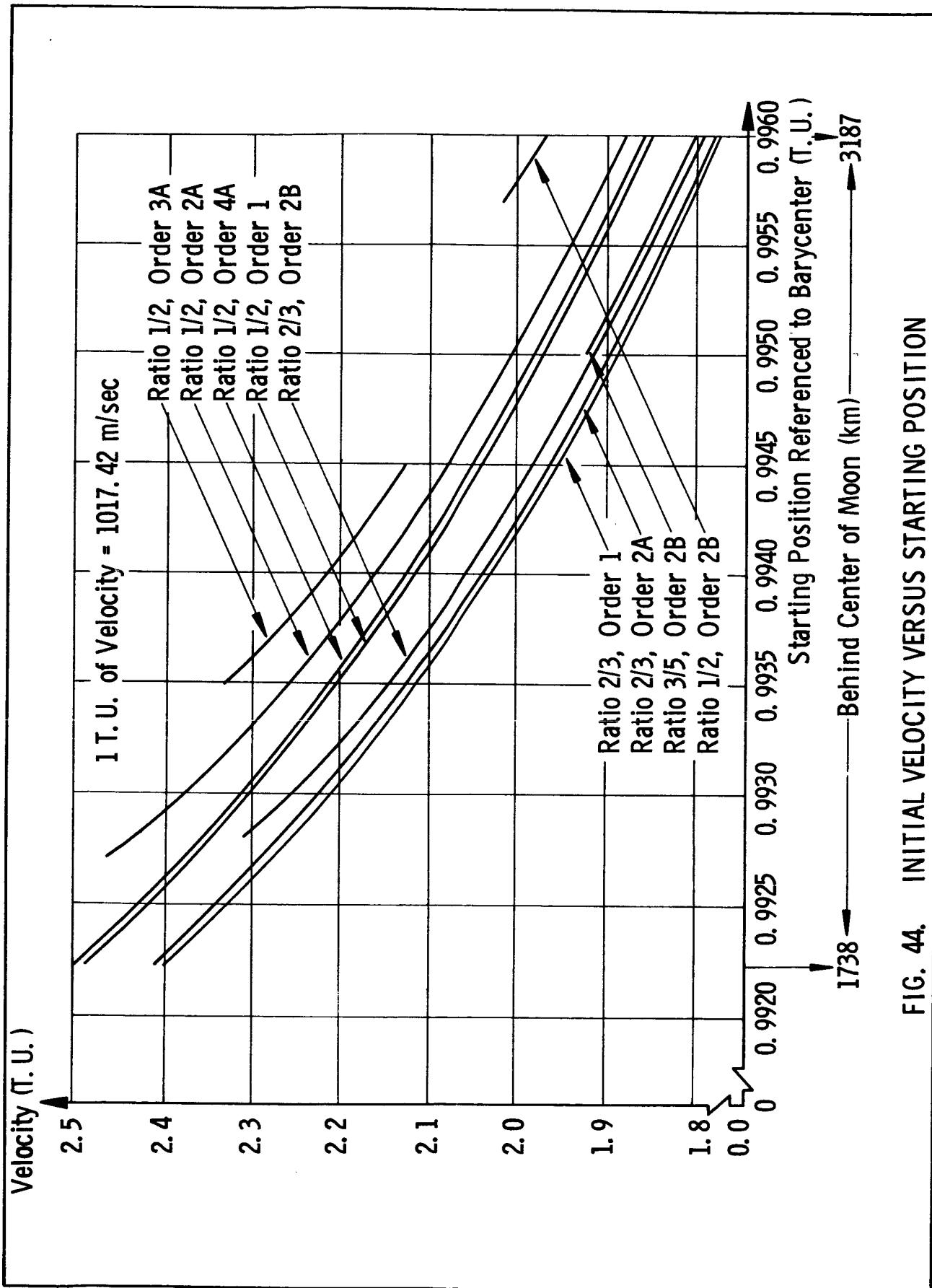
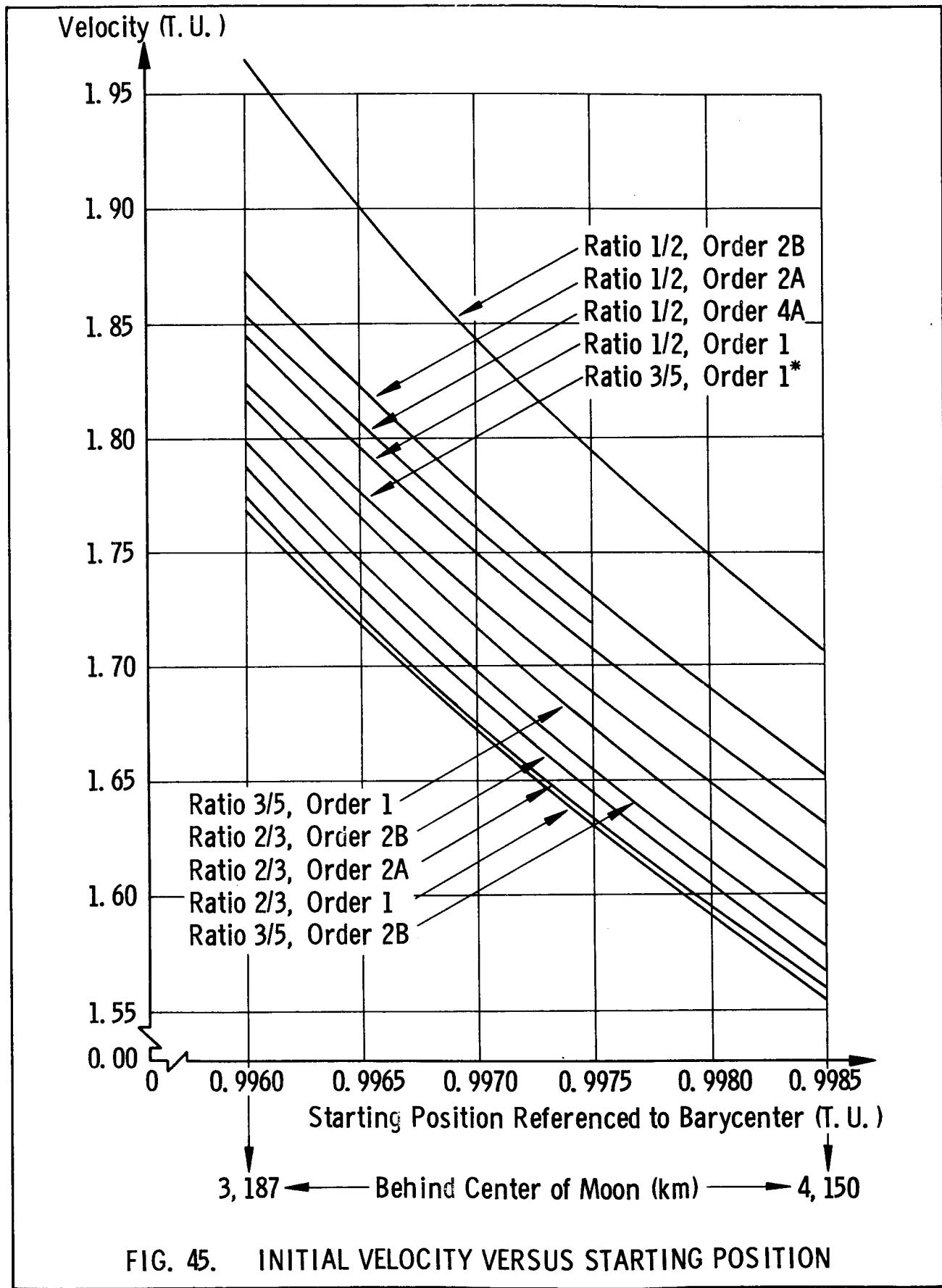


FIG. 44. INITIAL VELOCITY VERSUS STARTING POSITION



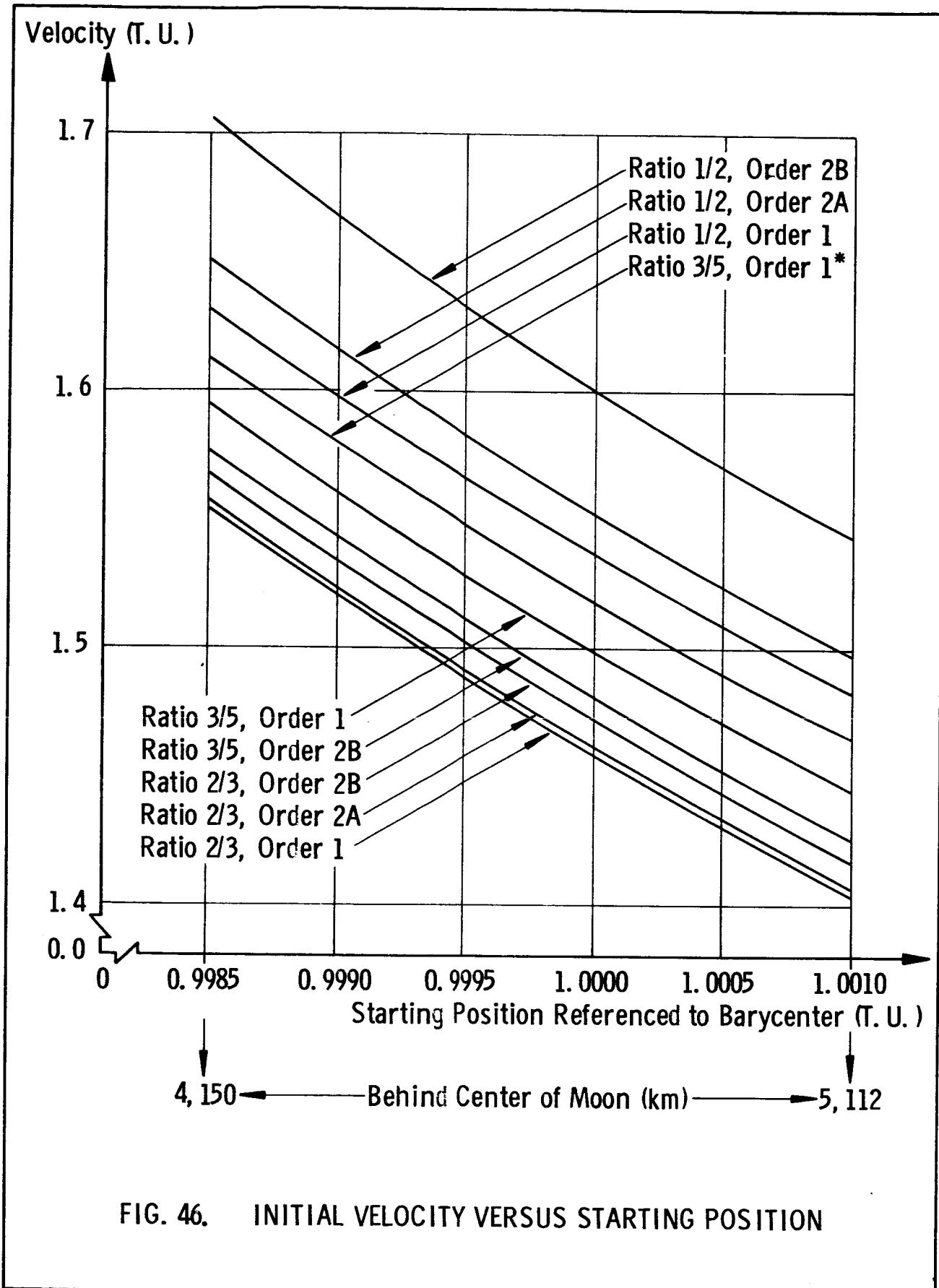
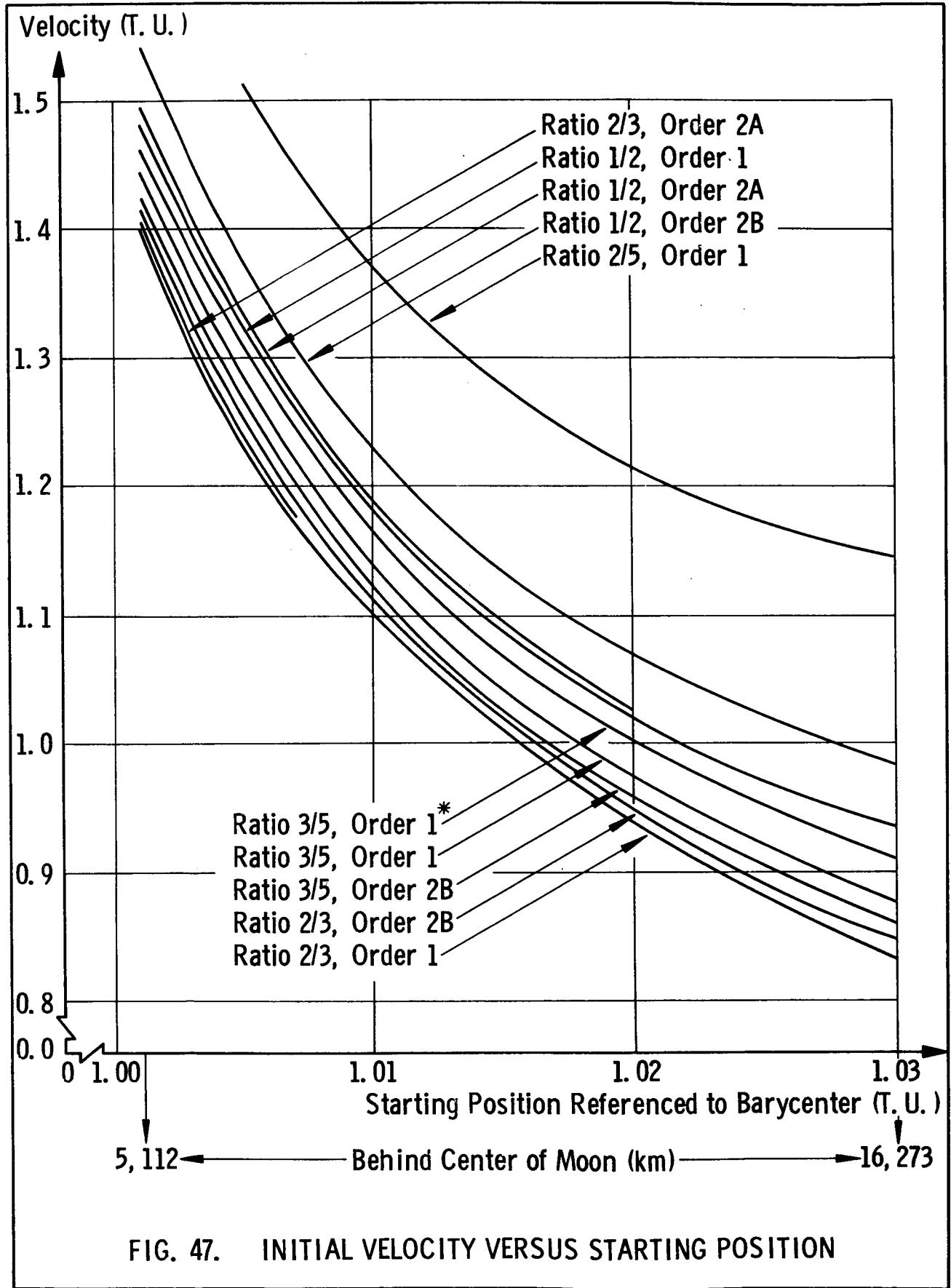


FIG. 46. INITIAL VELOCITY VERSUS STARTING POSITION



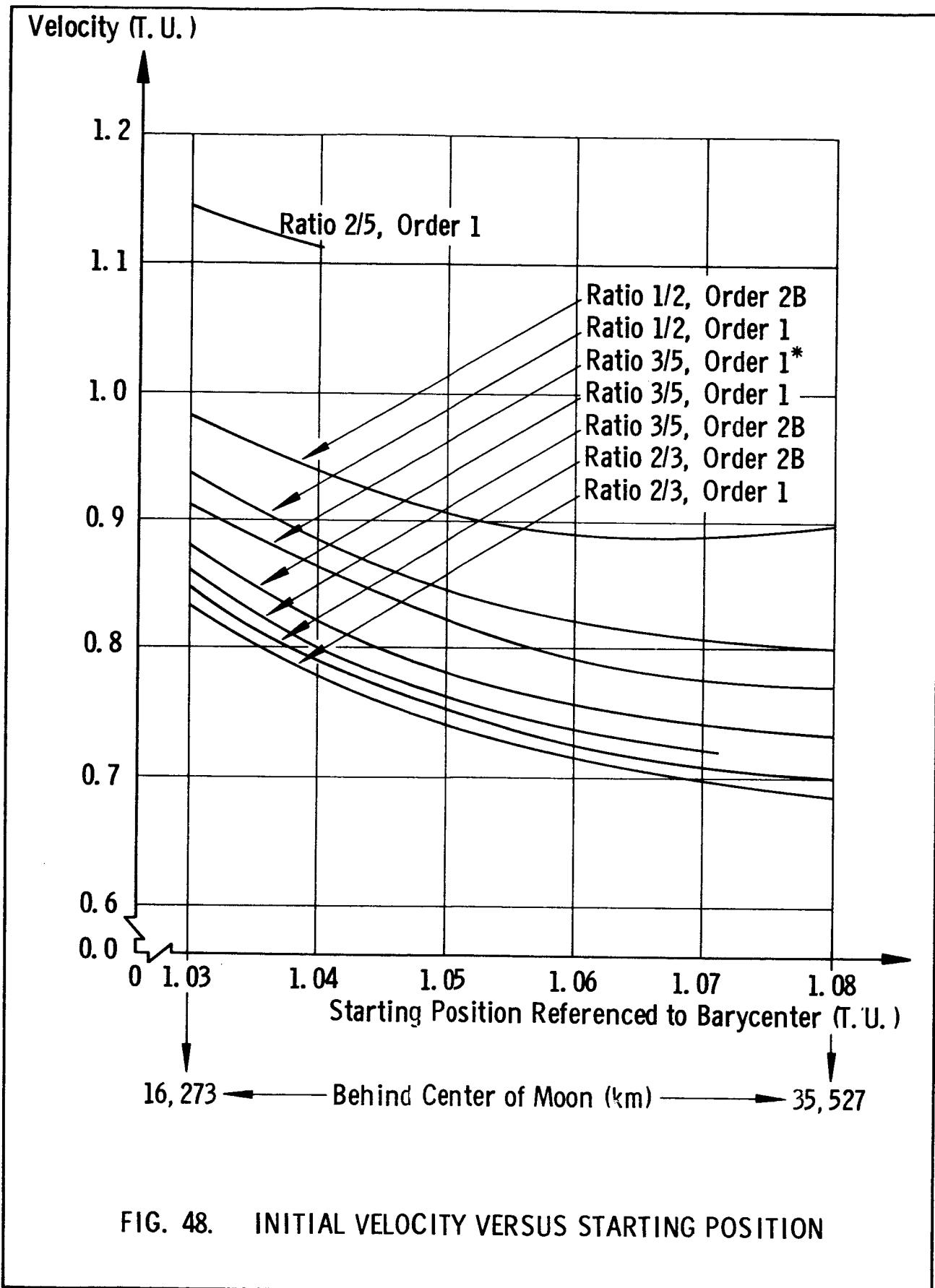


FIG. 48. INITIAL VELOCITY VERSUS STARTING POSITION

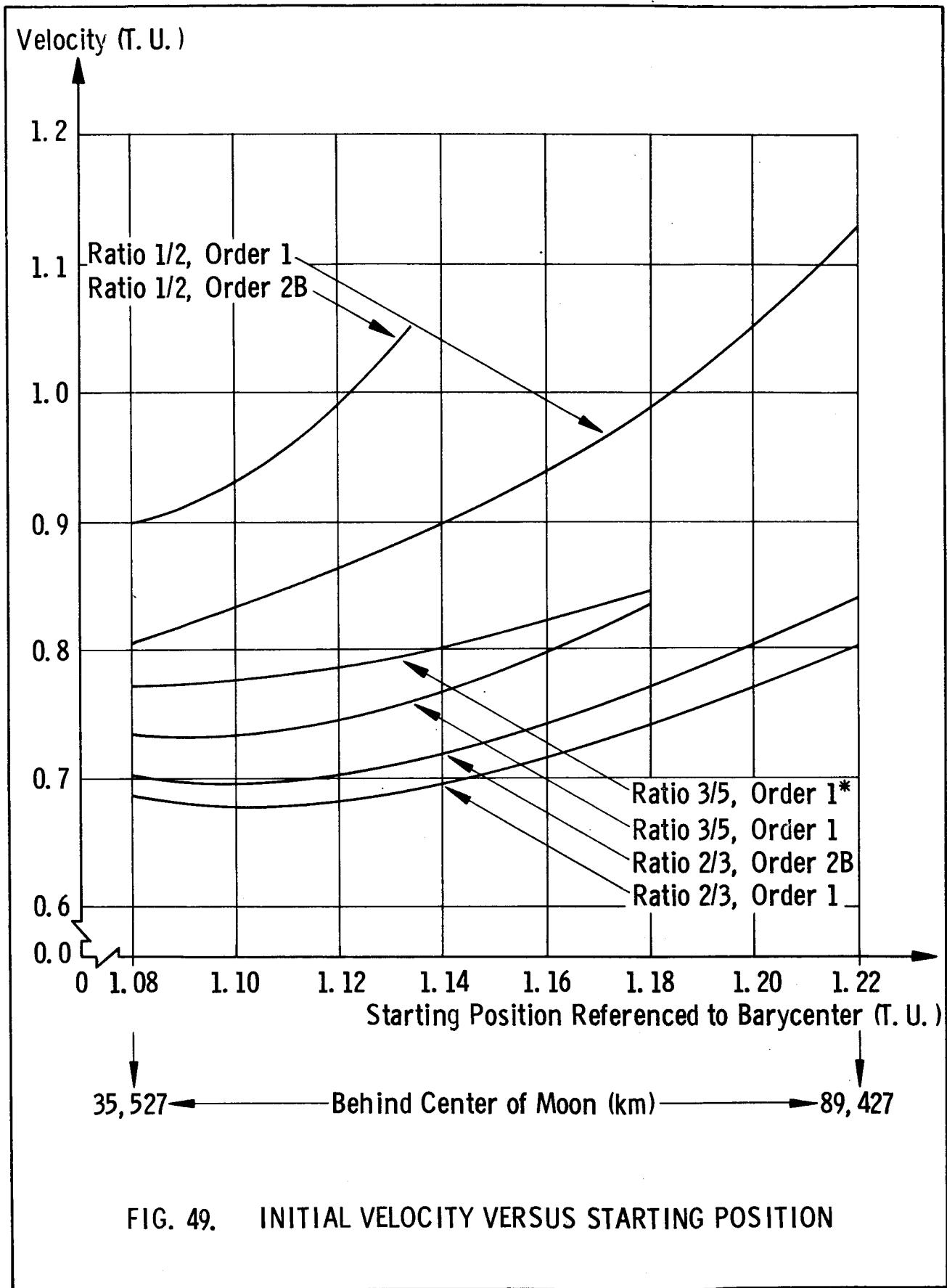


FIG. 49. INITIAL VELOCITY VERSUS STARTING POSITION

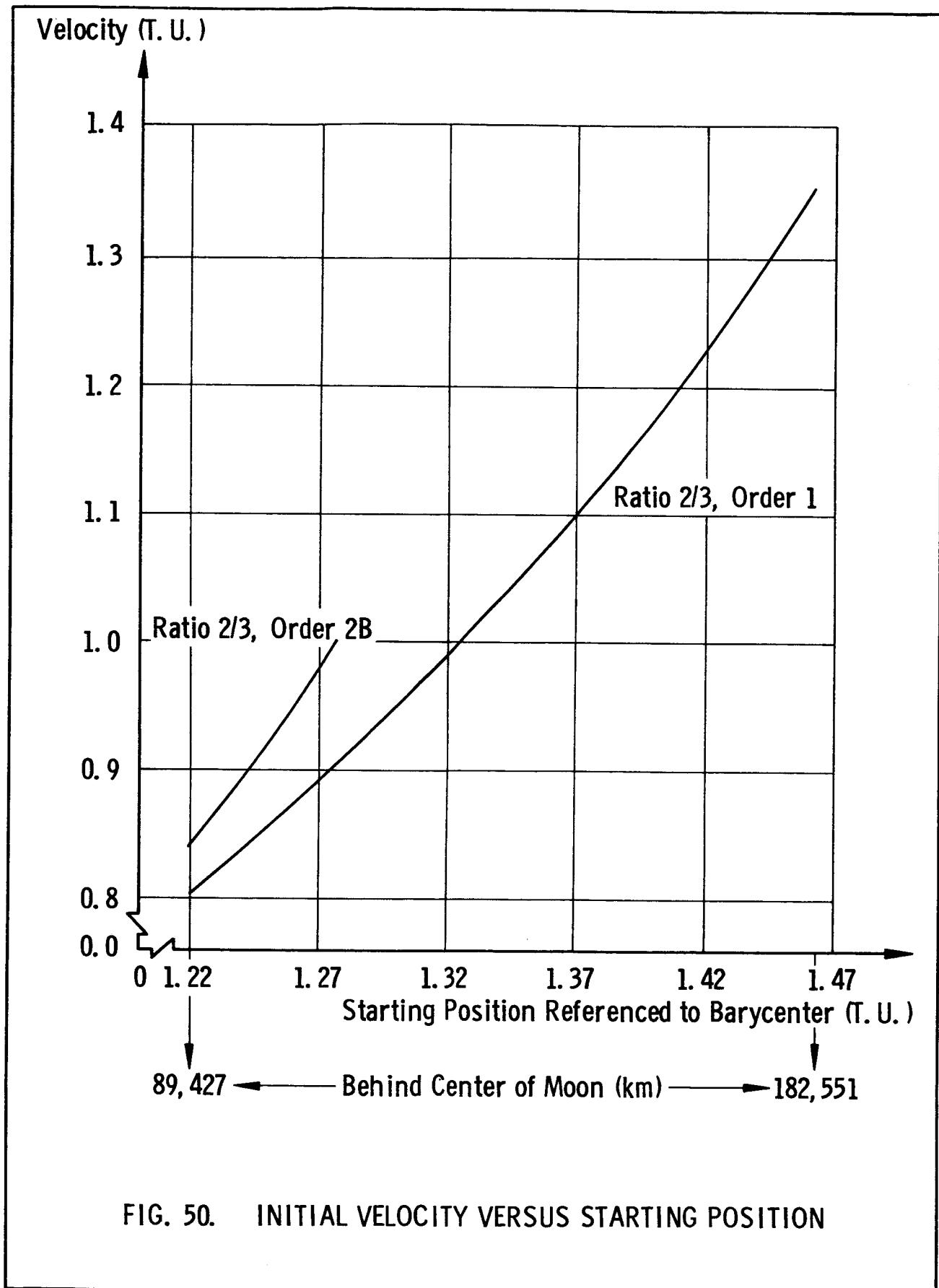


FIG. 50. INITIAL VELOCITY VERSUS STARTING POSITION

Ratio	Order	Class	First Approach to Moon (km)	Second Approach to Moon (km)	Close Approach to Earth (km)	Time From Earth to Moon (hr)	Time On Inner Leg (%)	Velocity (Space-Fixed) (m/sec)	Period (day)
1/2	1		89,080	-	6,552	155	47.5	1088.9	27.2
1/2	2	A	1,931	130,395	6,586	70	10.9	1086.7	53.3
1/2	2	B	3,081	3,341	6,560	76	11.6	1087.9	54.3
1/2	2	B	56,713	4,352	6,539	136	20.5	1089.3	55.2
2/3	1		182,551	-	6,574	208	31.5	1088.2	55.0
2/5	1		19,484	-	6,599	107	16.4	1083.6	54.4
1/2	3	A	2,317	91,507	6,577	72	7.5	1086.9	80.4

FIG. 51. ORBITS WITH INJECTION ALTITUDES  $\approx$  100 NAUTICAL MILES

August 17, 1964

APPROVAL

TM X-53108

CHARACTERISTIC FEATURES OF SOME PERIODIC ORBITS  
IN THE RESTRICTED THREE BODY PROBLEM

By

Wilton E. Causey

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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